

CELL DECOMPOSITION AND ODD CYCLES ON COMPACTIFIED RIEMANN'S MODULI SPACE

DOUGLAS J. LAFOUNTAIN AND R.C. PENNER

ABSTRACT. We introduce and investigate a bundle over the augmented Teichmüller space of a multiply punctured surface F by suitably decorating both punctures and double points of nodal surfaces; the total space of this bundle is shown to be equivariantly homotopy equivalent to the augmented Teichmüller space of F itself. We furthermore study a new bordification of the decorated Teichmüller space of F , which is shown to be equivalently homeomorphic to a real blow-up of its arc complex, by a space of *filtered screens* on the surface that arises from a natural elaboration of earlier work of McShane-Penner. An appropriate quotient of this space of filtered screens on F is shown to be equivariantly homotopy equivalent to its augmented Teichmüller space and to admit a natural cell decomposition, where cells are indexed by a suitable generalization of fatgraphs. The further quotient by the mapping class group gives the long-sought after cell decomposition of a space homotopy equivalent to the Deligne-Mumford compactification of the Riemann moduli space of F . As a first application of this technology, a plethora of odd-degree cycles is readily constructed, and we conjecture that these classes are homologically non-trivial.

1. INTRODUCTION

Let $F = F_g^s$ be a fixed surface of genus g with negative Euler characteristic and $s > 0$ punctures. In this paper, we will study the following diagram of spaces and maps:

$$\begin{array}{ccccccc} P\tilde{T}(F) & \hookrightarrow & \mathcal{FS}(F) & \xrightarrow{\psi} & \hat{T}(F) & \xrightarrow{\text{pr}} & \bar{T}(F) \\ & & \searrow \pi & \downarrow \chi & \swarrow \phi & & \\ & & & \mathcal{PG}(F) & & & \end{array}$$

where: $T(F)$ is the Teichmüller space of F , and $P\tilde{T}(F)$ is the projectivized decorated Teichmüller space obtained by decorating with horocycles the punctures of F modulo homothety; $\bar{T}(F)$ is the augmented Teichmüller

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space of F , which is a bordification of $T(F)$ obtained by adjoining strata corresponding to nodal surfaces. All of these spaces have been well-studied [2, 3, 21, 23].

The three new spaces are: the *space of filtered screens* $\mathcal{FS}(F)$, which is a cellular bordification of $P\bar{T}(F)$; the *decorated augmented Teichmüller space* $\hat{T}(F)$ which is a bundle over $\bar{T}(F)$ obtained by suitably decorating both punctures and double points of nodal surfaces; and the *space of punctured fatgraphs with partial pairing* $\mathcal{PG}(F)$, which will be obtained as a quotient of $\mathcal{FS}(F)$.

The mapping class group $MC(F)$ acts on all of these spaces, and all three of $P\bar{T}(F)$, $\mathcal{FS}(F)$, and $\mathcal{PG}(F)$ admit $MC(F)$ -invariant cell decompositions. Moreover, the map $\phi : \mathcal{PG}(F) \rightarrow \bar{T}(F)$ will be shown to be an $MC(F)$ -equivariant homotopy equivalence with the result that the space $\mathcal{PG}(F)/MC(F)$ is homotopy equivalent to the Deligne-Mumford compactification of Riemann's moduli space. This has been a long-standing open problem; for example, it is Problem 2 proposed by Dennis Sullivan in 2006 on the Centre for the Topology and Quantization of Moduli Spaces (CTQM) open problem list.

The stable cohomology of the Deligne-Mumford compactification $\bar{\mathcal{M}}(F)$ of Riemann's moduli space $\mathcal{M}(F) = T(F)/MC(F)$ was famously computed [10, 22] to be the polynomial algebra on the even-degree MMM classes [12, 15, 16]. On the other hand, the virtual Euler characteristic [5, 18] of $\mathcal{M}(F)$ shows that there are vast numbers of odd-degree classes, which are then necessarily unstable. With a direct construction in the space $\mathcal{PG}(F)/MC(F)$, we will exhibit many odd-degree cycles which we conjecture to be non-trivial in homology.

The cells $\mathcal{C}(G)$ in projectivized decorated Teichmüller space $P\tilde{T}(F)$ are indexed by isotopy classes of fatgraphs G in F , and the space of filtered screens $\mathcal{FS}(F)$ is obtained by adjoining to each $\mathcal{C}(G)$ new cells indexed by nested towers of recurrent subsets on G , where a subset of edges of a graph is recurrent if it has no univalent vertices and such a cell has codimension n in $\mathcal{C}(G)$ if it corresponds to n -fold nested recurrent subsets. Our motivation for defining $\mathcal{FS}(F)$ in this way arises directly from a comparison of the asymptotics of simplicial coordinates and lambda lengths for paths in $\mathcal{C}(G)$, namely, as described in the theorem of McShane-Penner (Theorem 3.5), the Filtered IJ Lemma (Lemma 3.18), and Proposition 3.21 and its corollaries. Using different terminology in the conformal setting [9], Looijenga defines a modified arc complex which also has cells indexed by nested towers of recurrent subsets on ribbon graphs G , but where the dimension of such cells is actually greater than that of $\mathcal{C}(G)$ and depends on the number of edges in each recurrent subset; we refer the reader to the Appendix in [14] for a treatment of this space. Thus, $\mathcal{FS}(F)$ is indeed different from the modified arc complex of Looijenga, both in its definition as well as in the precision

with which it captures asymptotic data for paths of vanishing simplicial coordinates.

Moreover, our main theorem (Theorem 5.24) involves a map to augmented Teichmüller space $\bar{T}(F)$ from a quotient of $\mathcal{FS}(F)$, namely, the space of punctured fatgraphs with partial pairing $\mathcal{PG}(F)$; this is important since it is $\mathcal{PG}(F)$ which is homotopy equivalent to $\bar{T}(F)$ and not $\mathcal{FS}(F)$. In contrast, the main theorem in [9] (see also the Appendix in [14]) involves a map directly from the modified arc complex to $\bar{T}(F)$ without a quotient; thus, our quotient space $\mathcal{PG}(F)$ is indeed new, even though partially paired punctured fatgraphs do appear in [9] as motivation for the modified arc complex. Furthermore, our proof that $\mathcal{PG}(F)$ is equivariantly homotopy equivalent to $\bar{T}(F)$ uses original techniques based on decorating $\bar{T}(F)$ using the combinatorial objects of nests on stratum graphs. Therefore, the resulting decorated augmented Teichmüller space $\hat{T}(F)$ is also new and solves the problem of distinguishing between decorated and undecorated punctures on the irreducible components of stable curves.

The structure of the paper is as follows: Because of the central importance of the topological space $\mathcal{PG}(F)$, we begin in the next paragraph with a description of the underlying set of objects that represent points in $\mathcal{PG}(F)$, which we will denote simply by non-calligraphic $PG(F)$. In section 2, we recall background and notation for the decorated Teichmüller theory of punctured surfaces. In section 3, we introduce and investigate the space of filtered screens and define its quotient space $\mathcal{PG}(F)$ thus topologizing $PG(F)$. In section 4, we define the decorated augmented Teichmüller space and show it is equivariantly homotopy equivalent to augmented Teichmüller space. Finally, in section 5, we establish our main theorems, namely, the existence of a cell decomposition for $\mathcal{PG}(F)$ (Theorem 5.15) and an equivariant homotopy equivalence from $\mathcal{PG}(F)$ to augmented Teichmüller space (Theorem 5.24). Section 6 describes our construction of odd-degree cycles with concluding remarks in section 7. We also include an appendix which describes the space of filtered screens as a blow-up of the arc complex.

1.1. The set $PG(F)$ of punctured fatgraphs. A *fatgraph* is a graph together with a cyclic ordering of the half-edges incident on each vertex, and a *punctured fatgraph* G is a fatgraph with certain of its vertices colored with a $*$ icon, which we call *punctured vertices*. A punctured fatgraph G determines a punctured surface $F(G)$ as follows. To each k -valent vertex of G is associated an oriented ideal k -gon, which is once-punctured if and only if the corresponding vertex is punctured. Sides of these polygons are identified in pairs, one such pair corresponding to each edge of G , so as to preserve orientations. We will refer to a puncture of the surface $F(G)$ as a *boundary component* of G if it does not correspond to a punctured vertex.

Definition 1.1. A *partial pairing* on a possibly disconnected punctured fatgraph is a collection of ordered pairs (α, β) where α is a punctured vertex and β is either another distinct punctured vertex or a boundary component

of a fatgraph, and where each punctured vertex or boundary component occurs at most once in the collection $\cup_{(\alpha,\beta)}\{\alpha,\beta\}$; see Figure 1. We will refer to α and β as *paired punctures*, and we denote the resulting *punctured fatgraph with partial pairing* by \bar{G} .

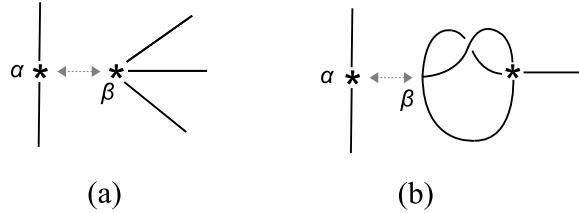


FIGURE 1. The two types of paired punctures.

Fix a surface F with $\chi(F) < 0$ and $s > 0$ punctures, where $\chi(F)$ denotes the Euler characteristic of F . A *curve family* σ in F is (the isotopy class of) a collection of curves disjointly embedded in F , no component of which is puncture-parallel or null-homotopic, and no two of which are parallel. Each such σ determines a nodal surface F_σ obtained by pinching each curve in σ to a double point or node and then separating the resulting nodes into pairs of punctures to produce a surface whose components F_1, \dots, F_n are called the *irreducible components* of F_σ .

Definition 1.2. We say that a punctured fatgraph with partial pairing \bar{G} is supported by F if the following hold:

- i. there are exactly s unpaired punctures;
- ii. $\chi(F(G_i)) < 0$ for all $i = 1, \dots, n$, where G_1, \dots, G_n are the punctured fatgraph components of \bar{G} ;
- iii. $\chi(F) = \sum_{i=1}^n \chi(F(G_i))$;
- iv. any two component fatgraphs are connected by a sequence of partial pairings, i.e., for any two distinct punctured fatgraph components G and G' , there is a sequence of components $G = G_1, G_2, \dots, G_m = G'$ such that G_ℓ has a puncture or boundary component paired with a puncture or boundary component of $G_{\ell+1}$ for each ℓ .

Such a punctured fatgraph with partial pairing \bar{G} determines a nodal surface $F(\bar{G})$, with the $F(G_i)$ as irreducible components and paired punctures corresponding to double points. Therefore, we consider *isotopy classes* of punctured fatgraphs with partial pairing \bar{G} supported by F , by which we mean that each \bar{G} carries a specification of both a curve family σ on F such that $F(\bar{G}) = F_\sigma$, as well as an isotopy class of fatgraph G_i for each irreducible component $F(G_i) = F_i$ of F_σ .

Moreover, we further consider *labeled* isotopy classes of punctured fatgraphs with partial pairing, which we still denote by \bar{G} , on which is specified a collection of positive weights, one weight for each edge, so that the sum

of weights on edges within each component is one. Two such labeled \bar{G} and \bar{G}' are therefore equivalent if and only if their respective curve families σ and σ' are equal, their component fatgraphs are isotopic, and the weights on their edges are identical. Equivalently, one imagines projectivizing weights separately on each component fatgraph.

Definition 1.3. For fixed $F = F_g^s$, the *set of punctured fatgraphs with partial pairing* for F , denoted $PG(F)$, is the set of all labeled isotopy classes of punctured fatgraphs with partial pairing \bar{G} supported by F and satisfying the following two conditions:

1. there exists a punctured fatgraph component G_i of \bar{G} such that all boundary components of G_i are unpaired;
2. for every finite sequence G_1, \dots, G_m of punctured fatgraph components such that G_ℓ has a boundary component paired with a puncture of $G_{\ell+1}$ for all ℓ , each fatgraph component G_i appears at most once, i.e., there are no cycles of paired boundary components.

A mapping class $\varphi \in MC(F)$ acts on $PG(F)$ via permutation of curve families and hence double points and paired punctures, and via push-forward of labeled isotopy classes of punctured fatgraphs on irreducible components. In particular, the pure mapping class group of each irreducible component of any F_σ acts on isotopy classes of punctured fatgraphs within that irreducible component.

In the course of our work, we will topologize the set $PG(F)$ to obtain the topological space $\mathcal{PG}(F)$ and exhibit a cell decomposition for $\mathcal{PG}(F)$ where cells are in one-to-one correspondence with isotopy classes of punctured fatgraphs with partial pairing, for which weights on edges in punctured fatgraph components serve as barycentric coordinates for the factor simplices which compose the cell. Such a cell decomposition will evidently be $MC(F)$ -invariant, and we will show that the quotient $\mathcal{PG}(F)/MC(F)$ is homotopy equivalent to the Deligne-Mumford compactification of moduli space.

2. BACKGROUND AND NOTATION

This section recalls the decorated Teichmüller theory of punctured surfaces from [11, 17] which is systematically treated among other topics in the monograph [21], to which we refer the reader for further detail.

2.1. Decorated Teichmüller Spaces. Let $F = F_g^s$ be a smooth oriented surface of genus g with $s > 0$ punctures such that $2g + s \geq 3$. The Teichmüller space $T(F)$ of F is comprised of all conjugacy classes of discrete and faithful representations $\rho : \pi_1(F) \rightarrow PSL_2(\mathbb{R})$ so that peripheral elements map to parabolics. The Fuchsian group $\Gamma = \rho(\pi_1(F)) < PSL_2(\mathbb{R})$ acts by isometries on the hyperbolic plane \mathbb{H} , and the induced hyperbolic structure on the surface is given by \mathbb{H}/Γ . We shall denote a point in Teichmüller space simply by (the conjugacy class of) a suitable Fuchsian group $\Gamma \in T(F)$.

The decorated Teichmüller space is the trivial bundle

$$\tilde{T}(F) = T(F) \times \mathbb{R}_{>0}^s,$$

where the fiber $\mathbb{R}_{>0}^s$ over a point $\Gamma \in T(F)$ is interpreted geometrically as an s -tuple of hyperbolic lengths of not necessarily embedded horocycles, one about each puncture. We shall denote a point in decorated Teichmüller space simply by $\tilde{\Gamma} \in \tilde{T}(F)$. More generally, we may likewise specify a non-empty collection \mathcal{P} of punctures and consider the \mathcal{P} -decorated Teichmüller space

$$\tilde{T}_{\mathcal{P}}(F) = T(F) \times \mathbb{R}_{>0}^{\mathcal{P}}.$$

Furthermore, the union

$$\tilde{T}_*(F) = \sqcup_{\mathcal{P} \neq \emptyset} \tilde{T}_{\mathcal{P}}(F)$$

inherits a topology from $T(F) \times \mathbb{R}_{\geq 0}^s$, where a horocycle is taken to be absent if it has length zero.

The mapping class group $MC(F)$ of isotopy classes of orientation-preserving homeomorphisms of F acts on $T(F)$ by push forward of metric and on $\tilde{T}(F)$ and $\tilde{T}_*(F)$ by push forward of metric and decoration, whereas only the subgroup of $MC(F)$ setwise fixing \mathcal{P} acts on $\tilde{T}_{\mathcal{P}}(F)$.

An *ideal arc* e in F is (the isotopy class of) an arc embedded in F with its endpoints at the punctures. A *quasi cell decomposition* or q.c.d. in F is a collection of ideal arcs pairwise disjointly embedded in F except perhaps at their endpoints, no two of which are parallel, so that each complementary component is either a polygon or an exactly once-punctured polygon. A q.c.d. is said to be *based* at the collection of punctures arising as endpoints of arcs comprising it; thus, there is exactly one punctured complementary region for each puncture at which a q.c.d. is not based. A q.c.d. is in particular an *ideal cell decomposition* or i.c.d. if each complementary region is unpunctured. A maximal i.c.d. (and q.c.d. respectively) has only complementary triangles (as well as once-punctured monogons) and is called an *ideal triangulation* (and a *quasi triangulation*).

2.2. Coordinates and Parameters. Associated to $\tilde{\Gamma} \in \tilde{T}(F)$ and an ideal arc e in F , we may assign a positive real number called its *lambda length* or *Penner coordinate* as follows: Straighten e to a Γ -geodesic connecting punctures in F , truncate this bi-infinite geodesic to a finite geodesic arc using the horocycles from $\tilde{\Gamma}$, let δ denote the signed hyperbolic length of this truncated geodesic (taken with positive sign if and only if the horocycles are disjoint), and set

$$\lambda(e; \tilde{\Gamma}) = \sqrt{\exp \delta}.$$

When $\tilde{\Gamma} \in \tilde{T}(F)$ is fixed or understood, we may write simply $\lambda(e)$.

Theorem 2.1. [Theorem 2.2.5 of [21]] *For any quasi triangulation E of F based at a set $\mathcal{P} \neq \emptyset$ of punctures, the natural mapping*

$$\Lambda_E : \tilde{T}_{\mathcal{P}}(F) \rightarrow \mathbb{R}_{>0}^E$$

given by

$$\tilde{\Gamma} \mapsto (e \mapsto \lambda(e; \tilde{\Gamma}))$$

is a real-analytic homeomorphism.

Remark 2.2. Lambda lengths are natural for the action of the mapping class group: if $\varphi \in MC(F)$ and $\varphi_*(\tilde{\Gamma})$ denotes push-forward of both metric and decoration, then

$$\lambda(e; \tilde{\Gamma}) = \lambda(\varphi(e); \varphi_*(\tilde{\Gamma}))$$

for any ideal arc e . The action of $MC(F)$ in coordinates can thus be computed and in fact gives prototypical examples of cluster varieties, cf. [21]; the diagonal lambda lengths e, f of a decorated ideal quadrilateral, which are said to be related by a “flip”, are themselves related to the frontier lambda lengths a, b, c, d in this cyclic order around the frontier by the Ptolemy relation $ef = ac + bd$.

There are a number of further parameters associated to a q.c.d. E of F and a point $\tilde{\Gamma} \in \tilde{T}(F)$ as follows. The hyperbolic length of a sub-arc of a decorating horocycle complementary to E is called its *h-length*. In particular for a decorated ideal triangle in F complementary to E , consider a subarc of the horocycle centered at one of its vertices subtended by two adjacent sides of the triangle as illustrated on the top in Figure 2, where the associated lambda lengths are indicated by lower-case a, b, c . The h-length α of this subarc is given by $\alpha = \frac{a}{bc}$; i.e., the opposite lambda length divided by the product of the adjacent lambda lengths. Likewise, in a once-punctured monogon with undecorated puncture and frontier with lambda length a as illustrated on the bottom of the same figure, the h-length α of the subarc of the decorating horocycle subtended by the frontier is given by $\alpha = \frac{2}{a}$. Of course, the h-length of any horocyclic arc is the sum of the h-lengths of sub-arcs comprising it; in particular, the hyperbolic length of the horocycle itself is such a sum. A convenient convention we shall continue to employ is that lower-case Roman letters refer to lambda lengths, and lower-case Greek letters refer to h-lengths.

For each i.c.d. E of F , there is a *dual fatgraph* $G(E)$, whose vertices and edges, respectively, are in one-to-one correspondence with components of $F - E$ and components of E ; an edge connects two vertices of $G(E)$ if its corresponding dual ideal arc lies in the common frontier of the corresponding complementary regions; the resulting graph $G(E)$ is a fatgraph since there is a cyclic ordering of half-edges incident on each vertex induced by the orientation of F . More generally for each q.c.d. E of F , there is a *dual (punctured) fatgraph* $G(E)$ defined analogously except that its vertices corresponding to once-punctured complementary regions are punctured vertices colored with a * icon. Examples are illustrated in Figure 3. In particular, vertices of $G(E)$ of valence one or two are necessarily punctured, and $G(E)$ has all its vertices trivalent or punctured univalent if and only if E is a quasi triangulation, in which case we say that $G(E)$ is *uni-trivalent*. Note that

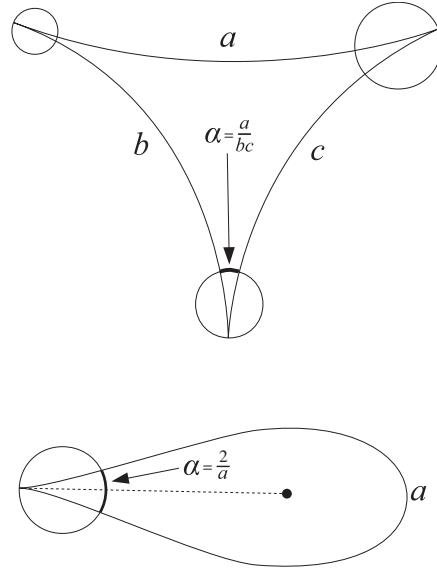


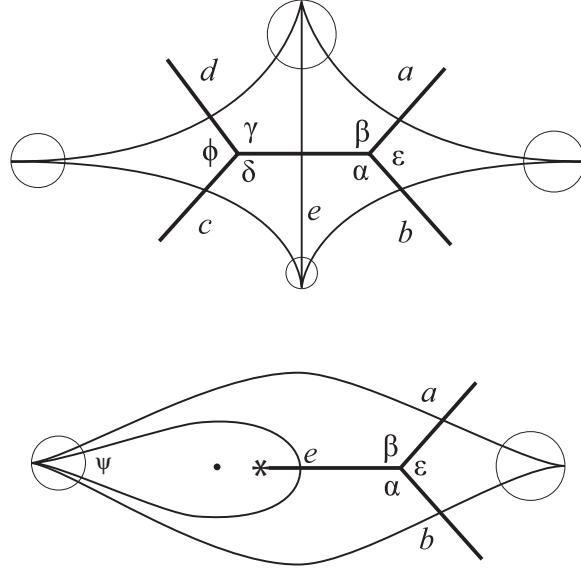
FIGURE 2. h-lengths in ideal triangles and once-punctured monogons.

any q.c.d. E' may be completed to a quasi triangulation E based at the same punctures, and $G(E')$ arises from $G(E)$ in this case by collapsing a sub-forest of edges of E , i.e., a collection of edges each component of which is contractible and contains at most a single punctured vertex.

Conversely, given a punctured fatgraph G , there is a corresponding q.c.d. $E(G)$ of its surface $F(G)$. Specifically, recall the polygons used in the construction of $F(G)$ in sub-section 1.1 and observe that the frontier edges of the polygons themselves give the associated q.c.d. $E(G)$ in $F(G)$.

Given $\tilde{\Gamma}$, lambda lengths on E induce an assignment of lambda lengths to dual edges on $G(E)$. Furthermore if E is a q.c.d., then h-lengths induce an assignment of included h-lengths to any two consecutive half-edges incident on a common vertex whether punctured or not. The top of Figure 3 introduces what will be our standard notation near an edge of $G(E)$ with distinct endpoints, where again nearby Roman letters denote lambda lengths and nearby Greek letters denote h-lengths. Analogously, the bottom of the same figure illustrates the geometry and standard notation near a once-punctured monogon. It will also sometimes be convenient to abuse notation and let the Roman letters denote at once an ideal arc in E , its dual edge in $G(E)$, or its lambda length for some specified $\tilde{\Gamma}$.

There is one further parameter of fundamental importance in the general theory and in particular for the considerations of this paper defined as follows. Given an assignment of lambda lengths to arcs in a quasi triangulation E and a fixed ideal arc $e \in E$ forming the common boundary of two complementary ideal triangles, the *simplicial coordinate* $X(e)$ associated to

FIGURE 3. Standard notation near an edge e .

e or its dual edge in $G(E)$ is given by:

$$X(e) = \frac{a^2 + b^2 - e^2}{abe} + \frac{c^2 + d^2 - e^2}{cde}$$

or in other words linearly in terms of h-lengths by:

$$X(e) = \alpha + \beta - \epsilon + \gamma + \delta - \phi,$$

where we are employing the notation on the top of Figure 3; likewise if e forms the common boundary of a complementary once-punctured monogon and ideal triangle, then

$$X(e) = \frac{a^2 + b^2 - e^2}{abe} + \frac{2}{e}$$

or in other words linearly in terms of h-lengths by:

$$X(e) = \alpha + \beta - \epsilon + \psi,$$

in the notation on the bottom of Figure 3.

Proposition 2.3. (Lemma 4.1.7 of [21]) *Suppose that e is an ideal arc in a q.c.d. E and complete E to two quasi triangulations E_1 and E_2 . Given lambda lengths on E_1 and E_2 that agree on E so that the simplicial coordinates on $E_1 - E$ and $E_2 - E$ vanish, the simplicial coordinate of e computed relative to E_1 agrees with that computed relative to E_2 .*

Proof. This non-trivial well-definedness of simplicial coordinates can be checked purely algebraically using the fact that finite sequences of flips act transitively on triangulations of a polygon and the Ptolemy relation. \square

We say that a cycle γ in a possibly punctured fatgraph is *quasi efficient* if it never consecutively traverses an edge followed by its reverse unless the intervening vertex is punctured.

Lemma 2.4 (Telescoping Lemma). *Suppose that γ is a quasi efficient cycle in a possibly punctured fatgraph consecutively traversing the edges e_1, \dots, e_n and let $\alpha_1, \dots, \alpha_n$ be the included h-lengths between consecutive edges along γ . Then*

$$\sum_{i=1}^n X(e_i) = 2 \sum_{i=1}^n \alpha_i.$$

Proof. This follows immediately from the formula for simplicial coordinates via local cancellation. \square

If a quasi efficient cycle γ in a fatgraph is a chain of distinct edges having only bivalent vertices, none of which is punctured, we will say that γ is a *simple cycle*.

It will often be convenient to projectivize the coordinates and parameters that have been defined, for which we introduce the following notation. For each $m \geq 0$, let $P\mathbb{R}_{\geq 0}^{m+1}$ denote the standard closed m -simplex in $\mathbb{R}_{\geq 0}^{m+1}$ with barycentric coordinates $t_i \geq 0$, for $i = 1, \dots, m+1$, satisfying $\sum_{i=1}^{m+1} t_i = 1$. Define *projectivized decorated Teichmüller spaces*

$$\begin{aligned} P\tilde{T}(F) &= \tilde{T}(F)/\mathbb{R}_{>0} \approx T(F) \times P\mathbb{R}_{>0}^s, \\ P\tilde{T}_{\mathcal{P}}(F) &= \tilde{T}(F)/\mathbb{R}_{>0} \approx T(F) \times P\mathbb{R}_{>0}^{\mathcal{P}}, \\ P\tilde{T}_*(F) &= \tilde{T}_*(F)/\mathbb{R}_{>0} \approx T(F) \times P\mathbb{R}_{\geq 0}^s. \end{aligned}$$

where in each case we decorate punctures with a *projectivized tuple* of hyperbolic lengths of all horocycles summing to two (corresponding to all simplicial coordinates summing to one).

It follows from Theorem 2.1 that projectivized lambda lengths give coordinates on the projectivized decorated Teichmüller spaces. Simplicial coordinates and h-lengths likewise descend to projective classes by homogeneity of their expressions in terms of lambda lengths. As a further point of notation, we may let $\lambda(e)$, $\alpha(e)$, and $X(e)$ denote lambda lengths, corresponding h-lengths, and simplicial coordinates on the edges e of some fixed q.c.d., with associated projectivized tuples $\bar{\lambda}(e)$, $\bar{\alpha}(e)$, and $\bar{X}(e)$, respectively.

2.3. Convex Hull Construction and Cell Decomposition. We turn finally to a discussion of the convex hull construction [4] in Minkowski 3-space (i.e., in \mathbb{R}^3 with the indefinite pairing given by $(x, y, z) \cdot (x', y', z') = xx' + yy' - zz'$) and the cell decomposition of decorated Teichmüller space [17]. Affine duality $u \mapsto h(u) = \{w \in \mathbb{H} : w \cdot u = -2^{-\frac{1}{2}}\}$, where $\mathbb{H} = \{w \in \mathbb{R}^3 : w \cdot w = -1\}$, identifies $u \in L^+ = \{u \in \mathbb{R}^3 : u \cdot u = 0\}$ with the collection of all horocycles $h(u)$ in the model \mathbb{H} of the hyperbolic plane.

Remark 2.5. In fact, the lambda length between two horocycles $h(u)$ and $h(v)$ is computed to be simply $\sqrt{-u \cdot v}$, so the identification with L^+ is geometrically natural. Furthermore, the simplicial coordinate can be computed to be $2^{\frac{3}{2}}abcd$ times the volume of the tetrahedron spanned by the vertices of a lift of a decorated quadrilateral to L^+ in the notation of Figure 3, where a vertex in L^+ corresponding to an undecorated puncture is taken to infinity in L^+ . Together with the fact that the Minkowski metric restricts to a Euclidean metric on any elliptic plane, one can give a more conceptual proof of Proposition 2.3.

A partially decorated structure $\tilde{\Gamma} \in \tilde{T}_{\mathcal{P}}(F)$ thus gives rise to a Fuchsian group Γ , realized as a subgroup of the component of the identity $SO^+(2, 1)$ of linear isometries of Minkowski space, as well as a Γ -invariant subset $\mathcal{B} \subset L^+$ corresponding to the partial decoration. It turns out that \mathcal{B} is a discrete set in $L^+ \cup \{0\}$, and its closed convex hull is a Γ -invariant convex body all of whose support planes are either elliptic or parabolic as conic sections, which we may imagine as a piecewise linear approximation to \mathbb{H} with its vertices in L^+ . The edges in the frontier of this convex body project in the natural way to a family $E(\tilde{\Gamma})$ of arcs connecting punctures in the underlying surface $F = \mathbb{H}/\Gamma$.

Theorem 2.6. (Theorem 5.5.9 of [21]) *For any non-empty set \mathcal{P} of punctures of F and any $\tilde{\Gamma} \in \tilde{T}_{\mathcal{P}}(F)$, the collection $E(\tilde{\Gamma})$ is a q.c.d. of F based at \mathcal{P} .*

Since the convex hull construction is invariant under homothety in Minkowski space, it is independent of global scaling of lengths of horocycles and therefore descends to the projectivized spaces. For any q.c.d. E of F , we are led to define

$$\begin{aligned} \mathcal{C}(E) &= \{\tilde{\Gamma} \in P\tilde{T}_*(F) : E(\tilde{\Gamma}) = E\} \\ &\cap \\ \bar{\mathcal{C}}(E) &= \{\tilde{\Gamma} \in P\tilde{T}_*(F) : E(\tilde{\Gamma}) \subseteq E\}, \end{aligned}$$

where equality and inclusion of q.c.d.'s are understood only up to proper isotopy fixing the punctures. Since the convex hull construction is natural in the sense that $E(\varphi_*(\tilde{\Gamma})) = \varphi(E(\tilde{\Gamma}))$ for any $\varphi \in MC(F)$ by construction, it follows that

$$\{\mathcal{C}(E) : E \text{ is a q.c.d. of } F\}$$

is an $MC(F)$ -invariant decomposition of $\tilde{T}_*(F)$.

Theorem 2.7. (Theorem 5.5.9 of [21]) *For any surface F of negative Euler characteristic with $s > 0$ punctures, the collection*

$$\{\mathcal{C}(E) : E \text{ is a q.c.d. of } F\}$$

of subsets is an $MC(F)$ -invariant ideal simplicial decomposition for $P\tilde{T}_(F)$. Indeed, for any quasi triangulation E , projectivized simplicial coordinates on*

arcs as barycentric coordinates give $\mathcal{C}(E)$ the natural structure of an open simplex in $P\tilde{T}_*(F)$, and this simplicial structure extends to $\bar{\mathcal{C}}(E)$ by adding certain faces, namely, those faces corresponding to sub-arc families of E with simplicial coordinates whose support is a q.c.d. of F .

Remark 2.8. In fact, a point in $\mathcal{C}(E)$ is determined by the lambda lengths on the edges of E , and these lambda lengths $\lambda(e_i)$ on the frontier edges e_i , for $i = 1, \dots, k$, of a complementary k -gon must satisfy the generalized triangle inequalities $\lambda(e_i) \leq \sum_{j \neq i} \lambda(e_j)$ with strict inequality if the k -gon is unpunctured. One can prove this directly algebraically using non-negativity of simplicial coordinates on edges of E or geometrically since a plane in Minkowski space containing three points in L^+ is elliptic or parabolic, respectively, if and only if the lambda lengths between them satisfy the strict triangle inequalities or some triangle equality.

Via the identification of fatgraphs with i.c.d.'s and punctured fatgraphs with q.c.d.'s, we shall sometimes equivalently regard the cells $\mathcal{C}(G) = \mathcal{C}(G(E))$ or $\bar{\mathcal{C}}(G) = \bar{\mathcal{C}}(G(E))$ as indexed by isotopy classes of fatgraphs embedded in F . In this context, the face relation is generated by collapsing an edge with distinct endpoints at most one of which is punctured.

3. SPACE OF FILTERED SCREENS

We begin by recalling the principal definitions and results from [11, 21] which are the starting point for the considerations of this paper and then proceed to the key new definitions and their investigation.

3.1. Screens. Consider a q.c.d. E of a surface F and its dual possibly punctured fatgraph $G = G(E)$. A subset $A \subset E$ determines a corresponding smallest subgraph $G(A)$ containing the edges A . Note that $G(A)$ may be disconnected and may have univalent or bivalent vertices which may be non-punctured. We say that A or $G(A)$ is *quasi recurrent* provided each univalent vertex of $G(A)$ is punctured. If $G(A)$ is quasi recurrent without punctured vertices, we simply say that $G(A)$ is *recurrent*¹.

Lemma 3.1. *A subset A of a q.c.d. is quasi recurrent if and only if for each $e \in A$ there is a quasi efficient cycle in $G(A)$ which traverses e .*

Proof. There can be no quasi efficient cycle traversing an edge incident on a univalent vertex which is not punctured. Conversely if there are no univalent vertices which are not punctured, arrange in a neighborhood of $G(A)$ in F to have two parallel copies of each edge which combine pairwise near the vertices in the unique manner, including at univalent punctured vertices, so that the result is a collection of simple closed curves. Each component curve

¹1-particle irreducible graphs from quantum field theory are those without univalent vertices that furthermore have no separating edges. These play a role in renormalization that is analogous to the role played here by recurrent fatgraphs.

of the result is quasi efficient, and for each edge, there is either one or two such curves traversing it by construction. \square

Corollary 3.2. *Any subset A of a q.c.d. contains a (possibly empty) maximal quasi recurrent subset given by the union of all quasi efficient cycles in $G(A)$.*

Definition 3.3. A screen \mathcal{A} on a q.c.d. E or its dual fatgraph $G(E)$ is a collection of subsets of E with the following properties:

- $E \in \mathcal{A}$;
- each $A \in \mathcal{A}$ is quasi recurrent;
- if $A, B \in \mathcal{A}$ with $A \cap B \neq \emptyset$, then either $A \subseteq B$ or $B \subseteq A$;
- for each $A \in \mathcal{A}$, we have $\bigcup\{B \in \mathcal{A} : B \subsetneq A\} \subsetneq A$.

Examples will be given in the next sub-section.

For each $A \in \mathcal{A}$, there is a maximal chain of proper inclusions $A \subset A^1 \subset \dots \subset A^n = E$, where each $A^i \in \mathcal{A}$, and we define the *depth* of A to be n . Each edge $e \in E$ has its *depth* defined to be $\max\{\text{depth of } A \in \mathcal{A} : e \in A\}$.

Indeed, each element $A \in \mathcal{A}$ other than $A = E$ has an immediate predecessor A' . We may furthermore consider the subsurface $F(G(A)) \subset F(G(A'))$, and define the *relative boundary* $\partial_{\mathcal{A}} A$ to be those boundary components of $F(G(A))$ which are neither puncture-parallel nor homotopic to a boundary component of $F(G(A'))$; moreover, if $G(A)$ includes a simple cycle component which is not a component in $G(A')$, we just include that simple cycle in $\partial_{\mathcal{A}} A$ (rather than two parallel copies arising from the boundary of the annular subsurface). Finally, the *boundary* of the screen itself is $\partial \mathcal{A} = \bigcup_{A \in \mathcal{A} - \{E\}} \partial_{\mathcal{A}} A$.

Definition 3.4. Suppose that $f_t : E \rightarrow \mathbb{R}_{>0}$, i.e., $f_t \in \mathbb{R}_{>0}^E$, for $t > 0$, is a continuous one-parameter family of functions on the edges of some q.c.d. E . Letting P denote projectivization, there is an induced $\bar{f}_t \in P(\mathbb{R}_{>0}^E)$, and compactness of $P\mathbb{R}_{\geq 0}^E \supset P(\mathbb{R}_{>0}^E)$ guarantees the existence of accumulation points of $\lim_{t \rightarrow \infty} \bar{f}_t$ in $P\mathbb{R}_{\geq 0}^E$. We say that f_t is *stable* provided there is a unique such limit point.

Theorem 3.5 (McShane-Penner, [11, 21]). *The cell $\mathcal{C}(G)$ in $P\tilde{T}_{\mathcal{P}}(F)$ corresponding to the fatgraph G contains a stable path of lambda lengths whose projection to $T(F)$ is asymptotic to a stable curve with pinch curves σ if and only if σ is homotopic to the collection of edge-paths $\partial \mathcal{A}$ for some screen \mathcal{A} on G .*

The various members of a screen \mathcal{A} correspond to subsets of edges whose lambda lengths diverge at least as fast as others, where $A \subset B \in \mathcal{A}$ if and only if the lambda lengths on edges in A diverge faster than those in B ; on the other hand, a screen does not keep track of the relative rates of divergence of disjoint elements. In fact, it is the triangle inequalities discussed in Remark 2.8 that obviously force the recurrence condition. The

difficult part of the proof of the previous theorem is estimating the lengths of the curves in $\partial\mathcal{A}$ in terms of lambda lengths.

3.2. Filtered screens. We next introduce a refinement of screens which plays the central role in this paper, where in contrast to screens, we keep track of rates of vanishing of simplicial coordinates along quasi recurrent subsets of a q.c.d., and we record relative rates of vanishing of disjoint subsets. We shall consider quasi recurrent subsets A of a q.c.d. E and their dual graphs $G(A)$, which as before may have bivalent vertices which are not punctured. Nevertheless, lambda lengths and simplicial coordinates of an edge $e \in A$ or $G(A)$ are defined to be those associated to e in the original fatgraph $G(E)$.

Definition 3.6. A *filtered screen of total level n* on a q.c.d. E or its dual fatgraph $G(E)$ is an ordered $(n+1)$ -tuple $\vec{\mathcal{E}} = (L^0, L^1, \dots, L^n)$ of pairwise disjoint and non-empty subsets of E so that:

- $\bigcup_{i=0}^n L^i = E$;
- for each $0 \leq k \leq n$, $L^{\geq k} = \bigcup_{i=k}^n L^i$ is quasi recurrent.

An arc $e \in L^k$ is said to have *level k* .

Notice that a filtered screen $\vec{\mathcal{E}}$ in particular determines its underlying q.c.d. $E = L^{\geq 0}$. Moreover, a filtered screen of total level zero on E is simply the q.c.d. $\vec{\mathcal{E}} = (E)$ treated as a 1-vector.

Definition 3.7. In analogy to screens, for each $0 \leq k < n$, we have the subsurface $F(G(L^{\geq k+1})) \subset F(G(L^{\geq k}))$ and define the $(k+1)$ -st *relative boundary* $\partial_{k+1}\vec{\mathcal{E}}$ to be those boundary components of $F(G(L^{\geq k+1}))$ which are neither puncture-parallel nor homotopic to a boundary component of $F(G(L^{\geq k}))$; if a component of $G(L^{\geq k+1})$ is a simple cycle which is not a component in $G(L^{\geq k})$, include just that cycle in $\partial_{k+1}\vec{\mathcal{E}}$. The *boundary* of $\vec{\mathcal{E}}$ itself is defined to be $\partial\vec{\mathcal{E}} = \bigcup_{i=1}^n \partial_i\vec{\mathcal{E}}$.

Given a filtered screen $\vec{\mathcal{E}}$, there is an associated screen

$$\mathcal{A}(\vec{\mathcal{E}}) = \{L^{\geq k} : 0 \leq k \leq n\}.$$

Although $\vec{\mathcal{E}} = \vec{\mathcal{E}'}$ if and only if $\mathcal{A}(\vec{\mathcal{E}}) = \mathcal{A}(\vec{\mathcal{E}'})$, not all screens arise in this way, only those which have a unique member at each depth. Conversely, to each screen \mathcal{A} on E is associated a filtered screen $\vec{\mathcal{E}}(\mathcal{A})$ where the level of each $e \in E$ is its depth; the resulting

$$L^{\geq k} = \{e \in E : \text{the depth of } e \text{ is at least } k\}$$

are quasi recurrent because if $e \in L^{\geq k}$, then e lies in a quasi recurrent subset $A \subseteq E$ of depth i for some $k \leq i \leq n$, whence also $A \subset L^{\geq k}$.

Insofar as $\partial\mathcal{A} = \partial\vec{\mathcal{E}}(\mathcal{A})$ and $\partial\vec{\mathcal{E}} = \partial\mathcal{A}(\vec{\mathcal{E}})$ by construction, we have the following reformulation of Theorem 3.5.

Corollary 3.8. *The cell $\mathcal{C}(G)$ in $P\tilde{T}_{\mathcal{P}}(F)$ corresponding to the fatgraph G contains a stable path of lambda lengths whose projection to $T(F)$ is asymptotic to a stable curve with pinch curves σ if and only if σ is homotopic to the collection of edge-paths $\partial\vec{\mathcal{E}}$ for some filtered screen $\vec{\mathcal{E}}$ on G .*

This discussion is somewhat misleading since we shall employ filtered screens to record rates of vanishing of simplicial coordinates rather than rates of divergence of lambda lengths as for screens, and these notions have yet to be related, cf. Lemma 3.14 and the results following it.

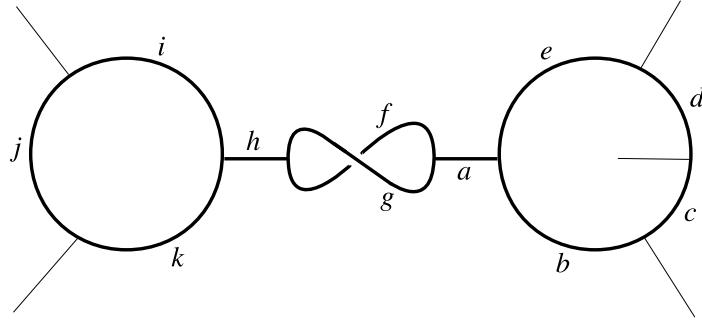


FIGURE 4. A sub-fatgraph of $G(E)$.

Example 3.9. Consider the sub-fatgraph of a fatgraph $G(E)$ depicted in Figure 4, and adopt the notation that “ $m-n$ ” means the set of labeled edges lexicographically between m and n and including m, n . The two screens

$$\mathcal{A} = \{E, a - k, a - g, i - k, f - g, b - e\}$$

and

$$\mathcal{A}' = \{E, a - k, (a - g) \cup (i - k), (f - g) \cup (b - e)\}$$

are distinct, however, $\vec{\mathcal{E}}(\mathcal{A}) = \vec{\mathcal{E}}(\mathcal{A}') = \vec{\mathcal{E}}$, where the filtered screen is

$$\vec{\mathcal{E}} = (E - (a - k), h, a \cup (i - k), (b - e) \cup (f - g)).$$

3.3. The space of filtered screens. In this section, we construct a simplicial complex whose simplices are indexed by filtered screens on q.c.d.’s based at \mathcal{P} for a fixed surface F . Let us adopt the notation that for filtered screens $\vec{\mathcal{E}}$ and $\vec{\mathcal{E}}'$, the respective sets of arcs of level k are denoted L^k and $(L')^k$.

Definition 3.10. Given a filtered screen $\vec{\mathcal{E}}$ of total level n , the *(open) cell associated to $\vec{\mathcal{E}}$* is a product of simplices defined as

$$\mathcal{C}(\vec{\mathcal{E}}) = P\mathbb{R}_{>0}^{L^0} \times P\mathbb{R}_{>0}^{L^1} \times \cdots \times P\mathbb{R}_{>0}^{L^n}$$

Definition 3.11. For any surface F , define the *space of filtered screens based at \mathcal{P}*

$$\mathcal{FS}_{\mathcal{P}}(F) = \left[\bigcup_{\substack{\text{filtered screens } \vec{\mathcal{E}} \text{ on some} \\ \mathcal{P}-\text{based q.c.d. of } F}} \mathcal{C}(\vec{\mathcal{E}}) \right] / \sim$$

where the face relation \sim is generated by the following:

- $\mathcal{C}(\vec{\mathcal{E}'})$ is a codimension-one face of $\mathcal{C}(\vec{\mathcal{E}})$ if there exists an edge $e \in L^k \subset E$ whose dual in $G(E)$ has distinct endpoints, at least one of which is non-punctured and disjoint from $G(L^{\geq k+1})$, and such that $(L')^k = L^k - \{e\}$. In other words, we pass from $\vec{\mathcal{E}}$ to $\vec{\mathcal{E}'}$ by removing such an arc from E ; alternatively, we collapse the dual edge in $G(E)$. Removing such an arc corresponds to setting the associated simplicial coordinate to zero in the factor simplex corresponding to L^k .
- $\mathcal{C}(\vec{\mathcal{E}'})$ is a codimension-one face of $\mathcal{C}(\vec{\mathcal{E}})$ if $\vec{\mathcal{E}'}$ arises from $\vec{\mathcal{E}}$ by combining two adjacent levels and shifting up by one all greater levels, i.e., we have

$$\vec{\mathcal{E}} = ((L')^0, (L')^1, \dots, (L')^{k-1}, (L')^k \cup (L')^{k+1}, (L')^{k+2}, \dots, (L')^n),$$

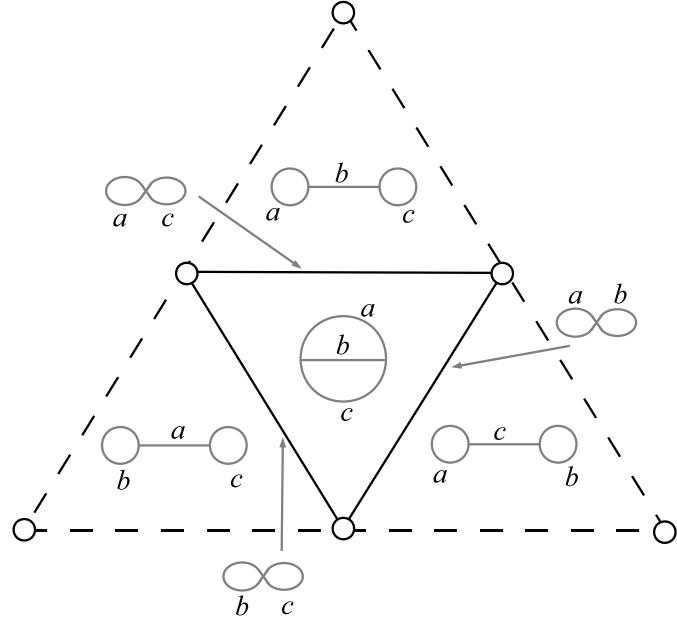
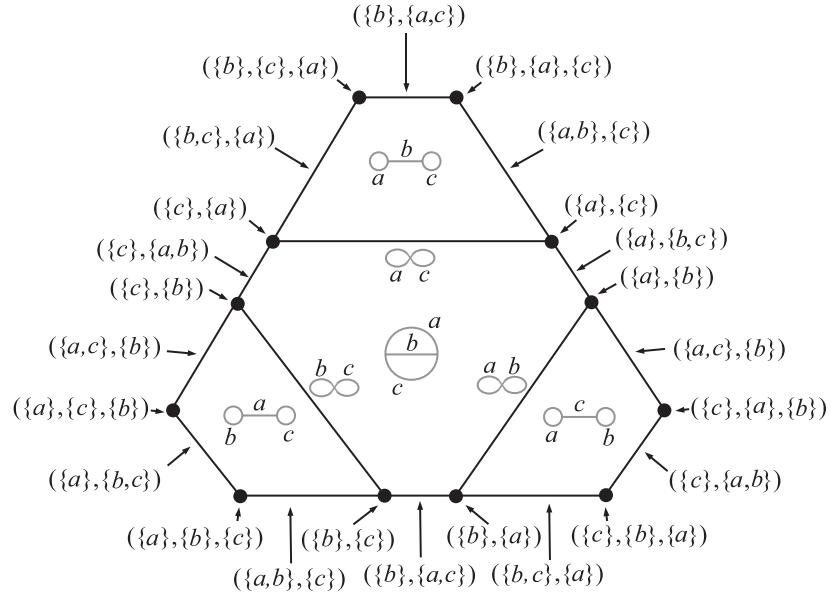
for some $0 \leq k < n$. In terms of coordinates, moving from $\mathcal{C}(\vec{\mathcal{E}})$ to $\mathcal{C}(\vec{\mathcal{E}'})$ corresponds to choosing a quasi recurrent subset $(L')^{\geq k+1}$ such that $L^{\geq k+1} \subsetneq (L')^{\geq k+1} \subsetneq L^{\geq k}$, and then allowing the simplicial coordinates of $(L')^{\geq k+1} - L^{\geq k+1}$ in $P\mathbb{R}_{>0}^{L^k}$ to go to zero all at comparable rates. In the limit, $(L')^k = L^{\geq k} - (L')^{\geq k+1}$ and $(L')^{k+1} = (L')^{\geq k+1} - L^{\geq k+1}$ are separately projectivized.

These face relations will be explicated in the sequel where we demonstrate that they precisely describe the asymptotics of stable paths constrained to lie in cells in \mathcal{P} -decorated Teichmüller space.

We will denote by $\mathcal{FS}(F)$ the space of filtered screens on i.c.d.'s of F for which all punctures are decorated, and we will refer to $\mathcal{FS}(F)$ simply as the *space of filtered screens*.

Example 3.12. Figure 5 depicts the cell decomposition of $P\widetilde{T}(F_0^3)$, where the small circles and dashed lines respectively represent zero- and one-simplices that are absent from $P\widetilde{T}(F_0^3)$. Figure 6 illustrates the space of filtered screens for F_0^3 , where the added faces are labeled by the corresponding filtered screens.

Remark 3.13. We observe that the cell decomposition for $\mathcal{FS}(F)$ is $MC(F)$ -invariant; moreover, $\mathcal{FS}(F)/MC(F)$ consists of finitely many cells, and therefore is a new compactification of Riemann's moduli space.

FIGURE 5. Projectivized decorated Teichmüller space for F_0^3 .FIGURE 6. The completion of projectivized decorated Teichmüller space for F_0^3 by the space of filtered screens.

3.4. Stable paths and the Filtered IJ Lemma. Fix a q.c.d. E of a surface F and consider its corresponding cell $\mathcal{C}(E) \subset P\tilde{T}_{\mathcal{P}}(F)$. To each point of $\mathcal{C}(E)$ is associated its tuple of projectivized simplicial coordinates and lambda lengths on the edges of E . By adding arcs if necessary to triangulate polygons complementary to E , we may and shall assume that E is a quasi triangulation, so that the associated fatgraph $G(E)$ is univalent. While the simplicial coordinates of the added arcs vanish, their lambda lengths do not.

We may thus regard a continuous (stable) path in $\mathcal{C}(E)$ as its associated continuous one-parameter family of projectivized simplicial coordinates $\bar{X}_t(e) \geq 0$ or lambda lengths $\bar{\lambda}_t(e) > 0$, defined for each edge $e \in E$ (and all $t \geq 0$). It will sometimes be convenient to de-projectivize these paths in various ways to produce corresponding one-parameter families $X_t(e) \geq 0$ and $\lambda_t(e) > 0$, which we may also regard as paths in the deprojectivized $\mathcal{C}(E)$ lying in $\tilde{T}_{\mathcal{P}}(F)$ itself.

Our primary immediate goal is to show that to each stable path in $\mathcal{C}(E)$, there is a canonical assignment of a point in a cell $\mathcal{C}(\vec{\mathcal{E}})$ associated with a filtered screen based on a q.c.d. (also based at \mathcal{P}) contained in E . This will be accomplished in the next sub-section after first here recalling and slightly extending several results from [21].

Lemma 3.14. *Given any point in a deprojectivized $\mathcal{C}(E)$ for any quasi triangulation E , the associated simplicial coordinates on E satisfy the no quasi vanishing cycle condition: there is no non-trivial quasi efficient cycle of edges in $G(E)$ each of whose simplicial coordinates vanishes.*

Proof. This follows from the fact that the convex hull construction of a \mathcal{P} -decorated surface produces a q.c.d. based at \mathcal{P} by Theorem 2.6. \square

Lemma 3.15. *Given any point in a deprojectivized $\mathcal{C}(E)$ and any arc e in the quasi triangulation E , the product of the lambda length of e and the simplicial coordinate of e is bounded above by four.*

Proof. If e is not incident on a punctured vertex, in the notation of Figure 3 we have

$$eX(e) = \frac{a^2 + b^2 - e^2}{ab} + \frac{c^2 + d^2 - e^2}{cd},$$

so each summand is twice a cosine by Remark 2.8 and the Euclidean law of cosines. If e is incident on a univalent punctured vertex in the notation of Figure 3, we have

$$eX(e) = \frac{a^2 + b^2 - e^2}{ab} + 2,$$

which is again bounded above by four. \square

Lemma 3.16. *Fix a point in a deprojectivized $\mathcal{C}(E)$ and suppose that a, b, e are the frontier edges of a triangle complementary to E with respective opposite h-lengths α, β, ϵ . If $1 \leq a, b, e$ and $\alpha \leq K < 1$, then we have $b, e \geq \frac{1}{2\sqrt{K}}$.*

Proof. We have $\alpha = \frac{a}{be} \leq K$ and $a \geq 1$ whence $K \geq \frac{a}{be} \geq \frac{1}{be}$, and so

$$(1) \quad \frac{\frac{1}{b}}{e} \leq K.$$

Furthermore, by Remark 2.8 we have $|b - e| < a$, and so dividing by be we find also

$$(2) \quad \left| \frac{1}{e} - \frac{1}{b} \right| < \frac{a}{be} \leq K.$$

It follows from inequality (1) that either $\frac{1}{b} \leq \sqrt{K}$ or $\frac{1}{e} \leq \sqrt{K}$. Without loss of generality concentrating on the former case and using inequality (2), we conclude

$$\frac{1}{e} \leq \frac{1}{b} + K \leq \sqrt{K} + K \leq 2\sqrt{K}$$

as was claimed. \square

Definition 3.17. Given a deprojectivized path λ_t in a deprojectivized $\mathcal{C}(E)$, and a subset $H \subset E$, define

$$I(H) = \{e \in H : \lim_{t \rightarrow \infty} \lambda_t(e) = \infty\}$$

and

$$J(H) = \{e \in H : \lim_{t \rightarrow \infty} X_t(e) = 0\}$$

The following result is of central importance in the next section and specializes to the so-called IJ Lemma of [11, 21] for $H = E$:

Lemma 3.18 (Filtered IJ Lemma). *For $G(E)$ uni-trivalent, let $H \subset E$, with $G(H)$ a quasi recurrent sub-fatgraph component that is not a simple cycle; that is, $G(H)$ may have bivalent vertices but has at least one trivalent vertex or univalent punctured vertex. Suppose that a deprojectivized path in $\mathcal{C}(E)$ has $\lambda_t(f) > 1$ for all $f \in G(H)$ and that each edge e of $G(E - H)$ incident on a bivalent vertex of $G(H)$ has $\lambda_t(e) \rightarrow 0$. Then $I(H) \subset J(H)$, and $G(I(H))$ is the maximal quasi recurrent sub-fatgraph of $G(J(H))$.*

Proof. If $\lambda_t(f) \rightarrow \infty$, then $X_t(f) \rightarrow 0$ by Lemma 3.15, so $I(H) \subset J(H)$.

To see that the maximal quasi recurrent sub-fatgraph of $G(J(H))$ is contained in $G(I(H))$, recall that a sub-fatgraph is quasi recurrent if and only if for each of its edges, it contains a quasi efficient cycle traversing the edge by Lemma 3.1. If γ is a quasi efficient cycle in $G(J(H))$, then $X_t(e) \rightarrow 0$ for each edge e traversed by γ by definition. According to the Telescoping Lemma 2.4, we have $\sum X_t(e) = 2 \sum \alpha_t(e)$, so the sum of the h-lengths included in γ goes to zero. If an edge e in γ is incident on a univalent punctured vertex, then since $\lambda(e) = 2/\alpha(e)$, where α is the included h-length, we see that $e \in G(I(H))$. Now, consider a vertex in γ with edges a, b, e incident upon it, where γ traverses b, e and includes the h-length α opposite a . In case all three of $a, b, e \in H$, then by Lemma 3.16, we have $\lambda_t(b), \lambda_t(e) \rightarrow \infty$ since $\alpha \rightarrow 0$ as required. It remains therefore to consider the case that $a \notin H$,

so that since $G(H)$ is not a simple cycle, b and e must lie in a linear chain of edges connecting vertices bivalent in H that begins and ends at vertices either trivalent or punctured univalent in H . Since all incident edges not in H along this chain have vanishing lambda lengths by assumption, all of the lambda lengths on the edges of this linear chain must have ratios asymptotic to unity by the triangle inequalities. Since the first and last edges in this chain must have lambda lengths which diverge by earlier remarks, in fact all of these edges must have lambda lengths which diverge. Thus, each edge traversed by γ lies in $G(I(H))$ as required.

For the reverse inclusion that $G(I(H))$ is contained in the maximal recurrent sub-fatgraph of $G(J(H))$, suppose that e is an edge with $\lambda_t(e) \rightarrow \infty$. As before, $X_t(e) \rightarrow 0$, so $e \in G(J(H))$. Thus, we need only show that there is a quasi efficient cycle in $G(J(F))$ traversing e . To this end, if the endpoints of e coincide, then there is such a cycle of length one. Otherwise by the triangle inequalities, there is an edge f adjacent to e at either of its non-puncture endpoints such that $\lambda_t(f) \rightarrow \infty$, so $X_t(f) \rightarrow 0$ by Lemma 3.15. Continuing in this manner, we find a quasi efficient cycle lying in $G(I(F))$ passing through e , which evidently also lies in $G(J(F))$ as desired. \square

3.5. Stable paths and filtered screens. We shall next canonically associate a point of $\mathcal{FS}_P(F)$ to a stable path in $\mathcal{C}(E) \subset P\tilde{T}_P(F)$.

Definition 3.19. Fix a stable path in $\mathcal{C}(E)$ with associated deprojectivized lambda lengths $\lambda_t(e)$ and simplicial coordinates $X_t(e)$, for $t \geq 0$ and $e \in E$. The lambda lengths of two edges $e, f \in E$ are said to be *comparable* if

$$0 < \lim_{t \rightarrow \infty} \frac{\lambda_t(e)}{\lambda_t(f)} < \infty,$$

be *incomparable* otherwise and be *asymptotically equal* provided

$$\lim_{t \rightarrow \infty} \frac{\lambda_t(e)}{\lambda_t(f)} = 1.$$

We say that the lambda length on e *diverges* if $\lambda_t(e) \rightarrow \infty$, *asymptotically vanishes* if $\lambda_t(e) \rightarrow 0$ and is *asymptotically bounded* if $1/L < \lambda_t < L$ for some $1 < L < \infty$ and all $t \geq 0$. The corresponding terms of asymptotic comparison among simplicial coordinates are defined analogously.

Definition 3.20. Suppose that X_t , for $t \geq 0$, is a deprojectivization which is bounded above of a stable path in $\mathcal{C}(E)$ for some q.c.d. E . Recursively define a nested collection $E = E_0 \supset E_1 \supset \dots \supset E_n$ of sets as follows. Fix t sufficiently large and choose some $e_k \in E_k$ so that $X_t(e_k)$ is greatest among edges in E_k ; define $E_{k+1} \subset E_k$ to be those edges that are not comparable to e_k , i.e., $E_k - E_{k+1}$ is comprised of those edges comparable to e_k . Thus, any two elements of $E_k - E_{k+1}$ are comparable, for $k = 0, \dots, n$, where we set $E_{n+1} = \emptyset$ for convenience. The collection $E_0 \supset E_1 \supset \dots \supset E_n$ is evidently invariant under overall scaling of simplicial coordinates and is called the *comparability filtration* on E induced by the stable path \bar{X}_t .

We come to our first important result about filtered screens:

Proposition 3.21. *Let E be a q.c.d. based at \mathcal{P} , and let \bar{X}_t be a stable path in the cell $\mathcal{C}(E) \subseteq \bar{\mathcal{C}}(E'') \subset P\tilde{T}_{\mathcal{P}}(F)$ corresponding to some quasi triangulation E'' of the surface F . Then there is a canonically determined limit point $p(\bar{X}_t) \in \mathcal{FS}_{\mathcal{P}}(F)$ in the cell $\mathcal{C}(\vec{\mathcal{E}}(\bar{X}_t))$ of a corresponding filtered screen $\vec{\mathcal{E}}(\bar{X}_t)$ on a \mathcal{P} -based q.c.d. contained in E with the following properties:*

- Two such paths \bar{X}_t and \bar{X}'_t in a common $\mathcal{C}(E)$ with respective comparability filtrations $E_0 \supset \dots \supset E_n$ and $E'_0 \supset \dots \supset E'_n$, which can be taken to have the same length by adjoining emptysets if necessary, have the same associated filtered screens $\vec{\mathcal{E}}(\bar{X}_t) = \vec{\mathcal{E}}(\bar{X}'_t)$ if and only if E_j and E'_j have a common (possibly empty) maximal quasi recurrent set R_j , and we have $R_j \cap (E_j - E_{j+1}) = R_j \cap (E'_j - E'_{j+1})$, for each $j = 0, \dots, n$.
- Moreover, the stable paths have the same associated points $p(\bar{X}_t) = p(\bar{X}'_t)$ if and only if they further have the same limiting ratios of simplicial coordinates on $R_j \cap (E_j - E_{j+1}) = R_j \cap (E'_j - E'_{j+1})$, for each $j = 0, \dots, n$.

Thus, there is a well defined mapping

$$\begin{aligned} \{\text{stable paths in } \mathcal{C}(E)\} &\rightarrow \mathcal{FS}_{\mathcal{P}}(F) \\ &\cup \\ \bar{X}_t &\mapsto p(\bar{X}_t) \in \mathcal{C}(\vec{\mathcal{E}}(\bar{X}_t)), \end{aligned}$$

where we must emphasize that the filtered screen $\vec{\mathcal{E}}(\bar{X}_t)$ is possibly not based on E but rather on some \mathcal{P} -based q.c.d. contained in E ; however, $p(\bar{X}_t) \in \mathcal{C}(\vec{\mathcal{E}}(\bar{X}_t))$ does not depend on the choice of quasi triangulation containing E . The filtered screen $\vec{\mathcal{E}}(\bar{X}_t)$ is called the *limiting filtered screen*, and the point $p(\bar{X}_t) \in \mathcal{FS}_{\mathcal{P}}(F)$ is called the *limiting point* of the stable path \bar{X}_t .

Proof. We shall algorithmically construct $p(\bar{X}_t) \in \mathcal{C}(\vec{\mathcal{E}}(\bar{X}_t))$ by repeated application of the Filtered IJ Lemma to partition the edges E'' of the quasi triangulation into nested collections of quasi recurrent subsets, while furthermore producing edges of E whose complement is the q.c.d upon which the filtered screen $\vec{\mathcal{E}}$ is based.

More precisely, at each stage $i \geq 0$ of the algorithm, we will have input given by a quasi recurrent collection $D_i \subseteq E''$ to which we apply the Filtered IJ Lemma in order to return as output four pairwise disjoint subsets whose union is D_i . Namely, the four disjoint subsets are:

- C_{i+1} containing arcs to be removed, i.e., whose dual edges in $G(E'')$ form a forest to be collapsed;

- Z_{i+1} containing arcs that will occur at some level of the filtered screen whose dual edges are disjoint simple cycles in $G(E)$ with comparatively divergent lambda lengths but slowly vanishing simplicial coordinates;
- B_{i+1} containing arcs whose appropriately scaled simplicial coordinates are bounded away from zero;
- D_{i+1} giving a quasi recurrent collection of arcs whose appropriately scaled simplicial coordinates are unbounded below and providing the input for the next stage of the algorithm.

The process terminates when $D_{i+1} = \emptyset$, say with $i = n$, and the desired filtered screen is defined in terms of the various constructed sets on the q.c.d. $E'' - \cup_{i=0}^n C_i \subseteq E$.

For the purposes of this proof, given a subset $H \subset E''$ such that the deprojectivized $X_t(f)$ vanishes for each $f \in H$, let us define

$$H^\infty = \{f \in H \mid \lim_{t \rightarrow \infty} \frac{X_t(f)}{\sup_{e \in H} X_t(e)} = 0\}.$$

In other words, H^∞ consists of those edges whose simplicial coordinates go to zero faster than the slowest in H . Note that when H is a simple cycle, each component of H^∞ is contractible.

To begin the construction, define $C_0 = Z_0 = B_0 = \emptyset$ and set $D_0 = E''$. Let us choose a deprojectivization λ_t of the stable path in lambda lengths corresponding to \bar{X}_t so that $\lambda_t(e) > 1$, for each $e \in D_0$. The Filtered IJ Lemma thus applies to $H = D_0 = E$ in E and yields $I(D_0) \subseteq J(D_0)$. We separate the sub-fatgraph $G(I(D_0))$ into its components and label the associated collections of edges I_1, \dots, I_m . In general, some of these components I_k correspond to simple cycles while others do not.

The general recursion is given by

$$C_{i+1} = [J(D_i) - I(D_i)] \cup \bigcup_{\text{cycles } I_k} I_k^\infty,$$

$$Z_{i+1} = \bigcup_{\text{cycles } I_k} (I_k - I_k^\infty)$$

$$B_{i+1} = D_i - J(D_i)$$

$$D_{i+1} = \bigcup_{\text{non-cycles } I_k} I_k$$

Let us pause to examine this output for the basis step $i = 0$. First of all, B_1 consists of all arcs in D_0 which have bounded simplicial coordinates and bounded lambda lengths for all t . As a result, the simplicial coordinates of the arcs in B_1 are all comparable and likewise their lambda lengths. We further observe that since any edges of $E'' - E$ have $\bar{X}_t = 0$, they are deepest in the comparability filtration, and hence not in B_1 .

Z_1 consists of the duals of edges in each simple cycle with divergent, asymptotically equal, lambda lengths but slowly vanishing simplicial coordinate; thus, components of $G(Z_1)$ consists of edges with comparable lambda lengths and comparable simplicial coordinates. Upon collapsing duals of $\cup_{\text{cycles}} I_k^\infty$, each component of $G(Z_1)$ again becomes a simple cycle. At the end of our procedure, we shall determine to which level of the filtered screen these cycles belong; again, we observe that edges of $E'' - E$ are not in Z_1 .

D_1 consists of the quasi recurrent I_k that are not simple cycles which have divergent lambda lengths and vanishing simplicial coordinates; this then becomes input for the next stage in the process. We emphasize that the edges of $G(D_1)$ have divergent lambda length compared to any adjacent edges not in $G(D_1)$.

Finally, C_1 consists of two subsets. The first, $J(F_0) - I(F_0)$, is comprised of those edges whose simplicial coordinates go to zero, but whose lambda lengths remain bounded; the duals of these edges will be appropriately collapsed at the end of the algorithm. The second subset arises from simple cycles with comparably divergent lambda lengths; the edges which are collapsed are linear chains whose simplicial coordinates go to zero faster than other simplicial coordinates in their cycle. Thus, these edges have comparable lambda lengths but vanishing simplicial coordinates and will also be appropriately collapsed at the end of the algorithm. To make sure that these collapses are legitimate, we have:

Claim $G(C_1)$ is a forest in $G(E)$ so that each component tree meets $G(D_1)$ in at most a single endpoint of an edge.

Proof. Indeed, each component of $G(J(D_0) - I(D_0))$ is a tree since $I(D_0)$ is the maximal quasi recurrent subset of $J(D_0)$ by the Filtered IJ Lemma. Each such tree can have at most one point in common with the $G(I_k)$ since if there were two such points of intersection, then there would be a quasi efficient cycle in $G(J(D_0))$ that was not in $G(I(D_0))$, contradicting the maximality of $I(D_0)$; this completes the proof of the claim.

Proceeding now to the general iteration, we are recursively provided with a quasi recurrent D_i so that $G(D_i)$ has no simple cyclic components and whose edges have divergent lambda lengths with respect to any adjacent edges of $G(E)$ not in $G(D_i)$. We perform an overall rescaling of *all of* the lambda lengths on E by the function $1/(\inf_{e \in D_i} \lambda_t(e))$. This in turn rescales all simplicial coordinates X_t by the reciprocal, and the projective simplicial coordinates \bar{X}_t of course remain unchanged. We then perform a second rescaling of lambda length of all edges in E by a constant to arrange that all the edges of D_i have scaled lambda lengths bigger than one. On the other hand, any edges incident to but not in $G(D_i)$ will have rescaled lambda lengths tending to zero. The hypotheses of the Filtered IJ Lemma are therefore satisfied, and we apply that lemma to D_i to obtain $I(D_i)$ and $J(D_i)$. As before, we may separate $I(D_i)$ into its components I_1, \dots, I_m and define a partition of D_i according to the recursion. The analysis of

the resulting sets B_{i+1} , Z_{i+1} , D_{i+1} , and C_{i+1} is entirely analogous to the previous discussion for the basis step; again we observe that edges of $E'' - E$ are not in B_{i+1} or Z_{i+1} , and we furthermore have:

Claim $G(\cup_{j=1}^{i+1} C_j)$ is a forest in $G(E)$ so that each component tree meets $G(D_{i+1})$ in at most a single endpoint of an edge.

Proof. To see this, we may assume by induction the previous claim that $G(\cup_{j=1}^i C_j)$ is a forest, and the only way components of $G(\cup_{j=1}^i C_j)$ can meet $G(D_i)$ is at a single endpoint of an edge. It follows that indeed $G(\cup_{j=1}^{i+1} C_j)$ is a forest, since $G(C_{i+1})$ is a forest in $G(D_i)$ by the Filtered IJ Lemma. Moreover, the component trees of $G(C_{i+1})$ can only possibly meet the new $G(I_k)$ at a single vertex, for otherwise, $I(D_i)$ could not be the maximal quasi recurrent subset of $J(D_i)$. As a result, if a component tree of $G(\cup_{j=1}^{i+1} C_j)$ were to meet the new $G(I_k)$ at two of its endpoints, only one of these could occur at a point where $G(C_{i+1})$ intersects the $G(I_k)$; the other must occur where $G(\cup_{j=1}^i C_j)$ intersects the $G(I_k) \subset G(D_i)$. This latter point plus the point where $G(\cup_{j=1}^i C_j)$ intersects $G(C_{i+1}) \subset G(D_i)$ would then represent two distinct points which $G(\cup_{j=1}^i C_j)$ has in common with $G(D_i)$, which by the inductive hypothesis cannot happen, concluding the proof of the claim.

Let us assume that the recursion terminates with $D_{n+1} = \emptyset$ having constructed the various subsets C_i, Z_i, B_i of E , for $i = 1, \dots, n$. We must next build the resulting filtered screen $\vec{\mathcal{E}} = \vec{\mathcal{E}}(\bar{X}_t)$ and identify the associated point $p = p(\bar{X}_t) \in \mathcal{C}(\vec{\mathcal{E}})$.

To this end, the collection $E' = E'' - \cup_{j=1}^n C_j \subseteq E$ is a q.c.d. according to the previous claim on which we shall define the filtered screen. The various sets Z_i, B_i , for $i = 1, \dots, n$ are pairwise disjoint, and the arcs in any such fixed set have comparable simplicial coordinates by construction. We may thus group together the arcs in $\cup_{i=1}^n (Z_i \cup B_i)$ into common comparability classes taking the union of these sets in a given class to define the edges at a fixed level in our filtered screen $\vec{\mathcal{E}}$ on E' . Quasi recurrence is assured from the construction, and the point $p \in \mathcal{C}(\vec{\mathcal{E}}(\bar{X}_t))$ is determined by limiting ratios of projective classes of simplicial coordinates in each level.

It follows by construction that p and $\vec{\mathcal{E}}$ have the characterizing properties in the statement of the proposition since the recursion itself depends only on the articulated data. In particular, since $E'' - E$ is a forest for any quasi triangulation E'' containing E , for any subset $A \subset E$, there is a one-to-one correspondence between quasi efficient cycles in $A \cup (E'' - E)$ and $A \cup (E_1'' - E)$ for any two quasi triangulations E'' and E_1'' containing E . Moreover, since the maximal recurrent subset of a set is simply the union of all quasi efficient cycles in that set, taking the maximal recurrent subset for each of $A \cup (E'' - E)$ and $A \cup (E_1'' - E)$ and then collapsing edges in $E'' - E$ and $E_1'' - E$ yields the maximal recurrent subset of A in either case. Thus, our construction does not depend on the choice of quasi triangulation. \square

In the notation of the preceding proof, there is a screen

$$\mathcal{A}(\bar{\lambda}_t) = \{E, I(D_0), I(D_1), \dots, I(D_n)\}$$

which is naturally associated to the stable path $\bar{\lambda}_t$ corresponding to \bar{X}_t , and we have the following corollary to the previous proof:

Corollary 3.22. *We have $\partial\vec{\mathcal{E}}(\bar{X}_t) = \partial\mathcal{A}(\bar{\lambda}_t)$, and by Lemma 4.38 in [11], these are precisely the curves whose hyperbolic lengths tends to zero.*

We now show in particular that each point of $\mathcal{FS}_{\mathcal{P}}(F)$ arises as the limiting point of some stable path in a cell of $P\tilde{T}_{\mathcal{P}}(F)$:

Proposition 3.23. *Given q.c.d.'s $E' \subseteq E$ and a point $p \in \mathcal{C}(\vec{\mathcal{E}}')$ for some filtered screen $\vec{\mathcal{E}}'$ on E' , there is a stable path \bar{X}_t in $\mathcal{C}(E)$ such that $p = p(\bar{X}_t)$ and $\vec{\mathcal{E}}(\bar{X}_t) = \vec{\mathcal{E}}'$.*

Proof. Suppose that $\vec{\mathcal{E}}' = (L^0, \dots, L^n)$ is of total level n and let $x_k(e)$ denote deprojectivized simplicial coordinates on $e \in L^k$ in the k th factor simplex $P\mathbb{R}_{>0}^{L^k}$ of $\mathcal{C}(\vec{\mathcal{E}}')$, for $k = 0, \dots, n$. Complete E to an ideal triangulation E'' and define for $t \geq 1$

$$X_t(e) = \begin{cases} 0, & \text{if } e \in E'' - E; \\ x_k(e) t^{-k}, & \text{if } e_i \in L^k, \text{ for } 0 \leq k \leq n; \\ t^{-(n+1)}, & \text{if } e \in E - E'. \end{cases}$$

The projectivization \bar{X}_t is a stable path in $\mathcal{C}(E)$ that by construction has the specified limiting point and filtered screen. \square

To conclude the discussion, we explain the face relations in Definition 3.11 in terms of comparability filtrations of stable paths. Let $\vec{\mathcal{E}} = (L^0, \dots, L^n)$ be a fixed filtered screen based on a q.c.d. $L^{\geq 0}$, and complete $L^{\geq 0}$ to a quasi triangulation E by adding a subset B of ideal arcs. We first observe as a consequence of Proposition 3.21 that stable paths \bar{X}_t in $\mathcal{C}(L^{\geq 0}) \subset \bar{\mathcal{C}}(E)$ having comparability filtration

$$E = (L^{\geq 0} \cup B) \supset (L^{\geq 1} \cup B) \supset \dots \supset (L^n \cup B) \supset B$$

are precisely those with $\vec{\mathcal{E}}(\bar{X}_t) = \vec{\mathcal{E}}$. In the following two corollaries of Proposition 3.21, we consider limiting filtered screens of stable paths in $\mathcal{C}(E)$ which slightly increase the comparability class for a proper subset of edges in a level L^k .

Corollary 3.24. *Let $\vec{\mathcal{E}} = (L^0, \dots, L^n)$ be a filtered screen based on a q.c.d. $L^{\geq 0}$, and complete $L^{\geq 0}$ to a quasi triangulation E by adding a subset B of ideal arcs. Let $e \in L^k$ be an arc whose dual edge in $G(E)$ has distinct endpoints, at most one of which is punctured. Consider any stable path \bar{X}'_t in $\mathcal{C}(E)$ having comparability filtration*

$$(L^{\geq 0} \cup B) \supset \cdots \supset (L^{\geq k} \cup B) \supset (L^{\geq k+1} \cup \{e\} \cup B) \supset (L^{\geq k+1} \cup B) \\ \supset \cdots \supset (L^n \cup B) \supset B.$$

Then we have the following:

1. if the dual edge of e in $G(L^{\geq 0})$ has a non-puncture endpoint disjoint from $G(L^{\geq k+1})$, then $\vec{\mathcal{E}}(\bar{X}'_t) = (L^0, \dots, L^{k-1}, L^k - \{e\}, L^{k+1}, \dots, L^n)$ is based on the q.c.d. $L^{\geq 0} - \{e\}$;
2. if the dual edge of e has both endpoints in $G(L^{\geq k+1})$ and $L^k - \{e\}$ is nonempty, then $\vec{\mathcal{E}}(\bar{X}'_t) = (L^0, \dots, L^{k-1}, L^k - \{e\}, \{e\}, L^{k+1}, \dots, L^n)$.

Proof. First observe that if the edge dual to e in $G(E)$ is contained in a simple cycle component I of L^k , then e necessarily satisfies the hypothesis of 1, and we also must have $L^k - \{e\}$ nonempty. Thus, when we rescale the deprojectivization X'_t of \bar{X}'_t in the k -th step of the recursion in Proposition 3.21 so that edges in $L^{\geq k} - (\{e\} \cup L^{\geq k+1})$ have bounded simplicial coordinates, we will have $I^\infty = \{e\}$. It follows that $\{e\} \in C_{k+1}$ is removed from the underlying q.c.d., and limiting filtered screen is then evidently $\vec{\mathcal{E}}(\bar{X}'_t) = (L^0, \dots, L^{k-1}, L^k - \{e\}, L^{k+1}, \dots, L^n)$ as required. As a result, we may assume for convenience that $L^{\geq k}$ has no simple cycle components.

For 1, we rescale at the k -th step of the recursion as above and observe that $J(L^{\geq k} \cup B) = \{e\} \cup L^{\geq k+1} \cup B$ and $L^{\geq k+1} \subset I(L^{\geq k} \cup B)$. However, $e \notin I(L^{\geq k} \cup B)$, for if there is a quasi efficient cycle in $\{e\} \cup L^{\geq k+1} \cup B$ passing through e , then all non-puncture vertices of e in $G(L^{\geq 0})$ would be in $L^{\geq k+1}$. Thus, $\{e\} \in C_{k+1}$ is removed from the underlying q.c.d. and limiting filtered screen, which is again $\vec{\mathcal{E}}(\bar{X}'_t) = (L^0, \dots, L^{k-1}, L^k - \{e\}, L^{k+1}, \dots, L^n)$. For this last statement, we remark that since $L^{\geq k}$ is itself quasi recurrent, there must in this case be another edge e' in L^k whose dual in $G(E)$ shares with the dual of e the endpoint disjoint from $G(L^{\geq k+1})$; thus, indeed $L^k - \{e\}$ is nonempty.

For 2, we again rescale at the k -th step and observe that in this case $\{e\} \cup L^{\geq k+1} \subset I(L^{\geq k} \cup B)$ since now $\{e\} \cup L^{\geq k+1}$ is recurrent. Moreover, since by hypothesis $L^k - \{e\}$ is nonempty, we have $B_{k+1} = L^k - \{e\}$. Furthermore, in the $(k+1)$ -st step of the recursion, the deprojectivization X'_t of \bar{X}'_t is rescaled so that e has bounded simplicial coordinate, and thus $B_{k+2} = \{e\}$, and the limiting filtered screen is $\vec{\mathcal{E}}(\bar{X}'_t) = (L^0, \dots, L^{k-1}, L^k - \{e\}, \{e\}, L^{k+1}, \dots, L^n)$. \square

Corollary 3.25. Let $\vec{\mathcal{E}} = (L^0, \dots, L^n)$ be a filtered screen based on a q.c.d. $L^{\geq 0}$, and complete $L^{\geq 0}$ to a quasi triangulation E by adding a subset B of ideal arcs. Let $A \subsetneq L^k$ be a proper subset with $A \cup L^{\geq k+1}$ quasi recurrent. If a stable path \bar{X}'_t in $\mathcal{C}(E)$ has comparability filtration

$$(L^{\geq 0} \cup B) \supset \cdots \supset (L^{\geq k} \cup B) \supset (L^{\geq k+1} \cup A \cup B) \supset (L^{\geq k+1} \cup B) \\ \supset \cdots \supset (L^n \cup B) \supset B.$$

then

$$\vec{\mathcal{E}}(\bar{X}_t') = (L^0, \dots, L^{k-1}, L^k - A, A, L^{k+1}, \dots, L^n).$$

Proof. The proof is identical to the reasoning for 2 in the above lemma, replacing $\{e\}$ with A . \square

3.6. The quotient space of filtered screens. To close this section, we shall describe the quotient of $\mathcal{FS}(F)$ which in section 5 will be shown to be equivariantly homotopy equivalent to the augmented Teichmüller space of F , as well as setwise identical to $PG(F)$. In fact, there is a labeled isotopy class of punctured fatgraph with partial pairing canonically associated to each point in $\mathcal{FS}_P(F)$ as follows:

Construction 3.26. Given a filtered screen $\vec{\mathcal{E}} = (L^0, \dots, L^n)$, consider its corresponding tower

$$L^{\geq 0} \supset L^{\geq 1} \supset \cdots \supset L^{\geq n}$$

of quasi recurrent sets. There is a corresponding tower of inclusions

$$G(L^{\geq 0}) \supset G(L^{\geq 1}) \supset \cdots \supset G(L^{\geq n}),$$

where each component of $G(L^{\geq k})$, for each $k = 0, 1, \dots, n$, is either a horocycle, a simple cycle, or a fatgraph. The three possibilities for a component of $G(L^{\geq k+1})$ in $G(L^{\geq k})$ are illustrated in bold stroke in Figure 7, together with a modification of $G(L^{\geq k})$ to be performed in each case in order to produce a partial pairing indicated by double arrows on the resulting punctured fatgraph. These modifications are performed in order of decreasing level beginning with $G(L^{\geq n}) \subset G(L^{\geq n-1})$ so as to produce an isotopy class of punctured fatgraph $\bar{G}(\vec{\mathcal{E}})$ with partial pairing associated to $\vec{\mathcal{E}}$.

Furthermore, given a point $p \in \mathcal{C}(\vec{\mathcal{E}})$, there are associated projectivized simplicial coordinates induced on the edges of each component punctured fatgraph of $\bar{G}(\vec{\mathcal{E}})$ as follows: First, separately deprojectivize each factor simplex in $\mathcal{C}(\vec{\mathcal{E}})$ in order to have coordinates defined on each edge.

- For edges arising from operations (a) or (b) in Figure 7, take the same coordinate in $\bar{G}(\vec{\mathcal{E}})$.
- For edges arising from operation (c) in Figure 7, take as coordinate the sum of the coordinates of the constituent edges that have combined to form the new edge.

Finally, projectivize these coordinates on each punctured fatgraph component of $\bar{G}(\vec{\mathcal{E}})$ to obtain projective simplicial coordinates on each component of $\bar{G}(\vec{\mathcal{E}})$.

The procedure for operation (c) is justified by the following calculation: Figure 8 illustrates the included h-lengths near a consecutive sequence of

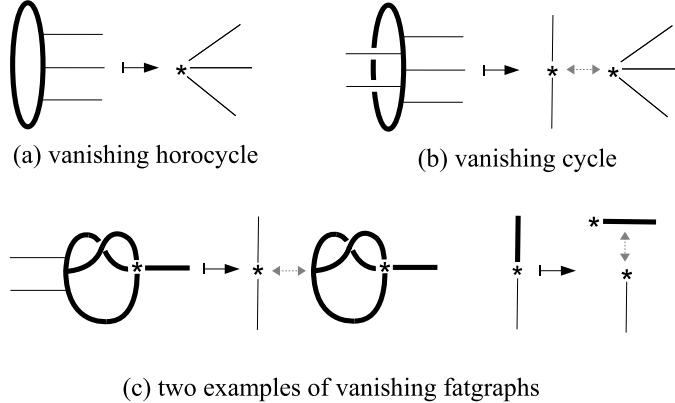


FIGURE 7. The three operations performed on a filtered screen on a fatgraph to obtain the corresponding punctured fatgraph with partial pairing.

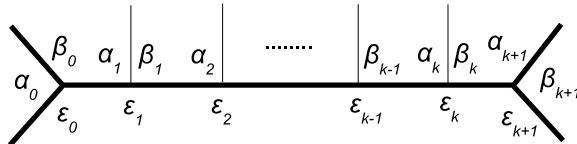


FIGURE 8. Edges with bounded simplicial coordinates incident on a sub-fatgraph whose lambda lengths diverge.

edges with bounded simplicial coordinates adjacent to a linear chain of edges in bold strokes with vanishing simplicial coordinates. Using the expression for simplicial coordinates as a linear combination of included h-lengths, one finds that the sum of simplicial coordinates along the linear chain is given by

$$\sum_{i=1}^{k+1} X(e_i) = \beta_0 + \epsilon_0 - \alpha_0 + 2 \sum_{i=1}^k \epsilon_i + \alpha_{k+1} + \epsilon_{k+1} - \beta_{k+1}$$

The term $2\sum_{i=1}^k \epsilon_i$ goes to zero, thus leaving the expression for the usual simplicial coordinate of the single edge arising from the linear chain in keeping with the specification of simplicial coordinates above; a similar calculation holds if the linear chain terminates at a punctured vertex.

Example 3.27. The top of Figure 9 illustrates the fatgraph $G(E)$ treated in Example 3.9 on which we consider the two filtered screens

$$\vec{\mathcal{E}} = (E - [(a-g) \cup (i-k)], (a-g), (i-k)),$$

$$\vec{\mathcal{E}'} = (E - [(a-g) \cup (i-k)], (i-k), (a-g))$$

in the earlier notation. Assuming that $i - k$ is a horocycle, the common punctured fatgraph with pairing $\bar{G}(\vec{\mathcal{E}}) = \bar{G}(\vec{\mathcal{E}}')$ is illustrated on the bottom

in Figure 9, where the deprojectivized simplicial coordinates are indicated by capital Roman letters.

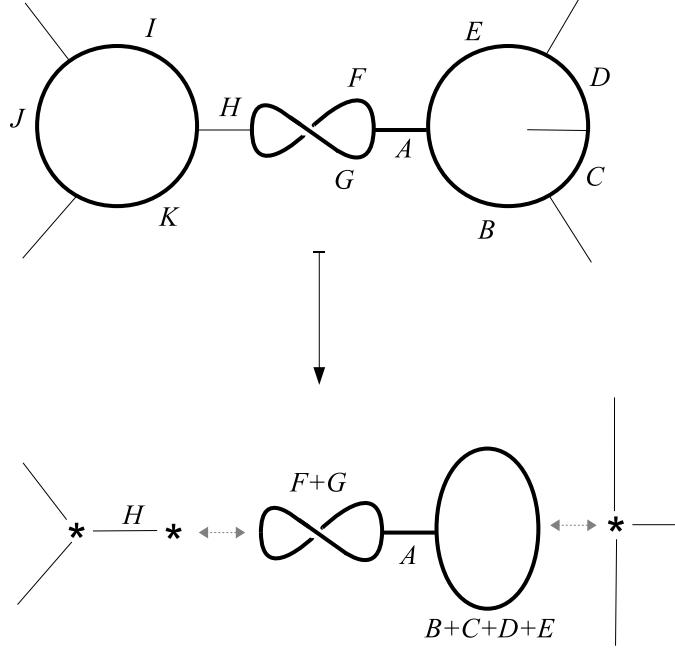


FIGURE 9. Example of Construction 3.26.

Definition 3.28. Given $p \in \mathcal{C}(\vec{\mathcal{E}}) \subset \mathcal{FS}_P(F)$, the isotopy class of punctured fatgraph $\bar{G}(\vec{\mathcal{E}})$ with partial pairing together with the labeled assignment of projective simplicial coordinates on each component punctured fatgraph determined by Construction 3.26 is denoted $\pi(p)$. Two points $p, q \in \mathcal{FS}_P(F)$ are *equivalent* and we write $p \sim q$ if $\pi(p) = \pi(q)$ with quotient space $\mathcal{FS}_P(F)/\sim$.

Definition 3.29. Define the P -decorated space of punctured fatgraphs with partial pairing for F , denoted by $\mathcal{PG}_P(F)$, by the following: a labeled isotopy class of punctured fatgraph with partial pairing \bar{G} lies in $\mathcal{PG}_P(F)$ if and only if $\bar{G} = \pi(p)$ for some point $p \in \mathcal{C}(\vec{\mathcal{E}}) \subset \mathcal{FS}_P(F)$. There is thus a bijection between $\mathcal{PG}_P(F)$ and $\mathcal{FS}_P(F)/\sim$, and we topologize $\mathcal{PG}_P(F)$ with the quotient topology via this bijection.

In the case of $\mathcal{FS}(F)/\sim$ where P is the set of all punctures of F , we simply denote this space by $\mathcal{PG}(F)$, and refer to it as the *space of punctured fatgraphs with partial pairing* for F .

In the course of our work, we will show that $\mathcal{PG}(F) = PG(F)$ as a set. An example of $\mathcal{PG}(F_0^3)$ is shown in Figure 10. In Section 5, we will describe a $MC(F)$ -invariant cell decomposition for $\mathcal{PG}(F)$ and show

that $\mathcal{PG}(F)/MC(F)$ is equivariantly homotopy equivalent to the Deligne-Mumford compactification of moduli space. To do so, however, we first need to decorate augmented Teichmüller space using new objects termed “nests on stratum graphs”.

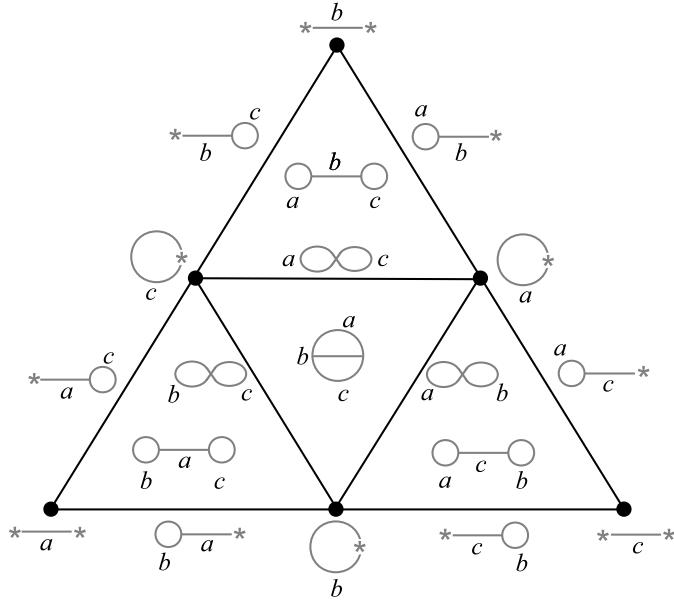


FIGURE 10. $\mathcal{PG}(F_0^3)$ and its cell decomposition.

4. DECORATED AUGMENTED TEICHMÜLLER SPACE

4.1. Augmented Teichmüller space. Closely following the treatments [3, 23], this section recalls the augmented Teichmüller space $\bar{T}(F)$ of a surface $F = F_g^s$, introduced in [2], which provides a bordification of $T(F)$.

A maximal curve family $\mathcal{P} = \{\alpha_i\}_1^N$, where the interior of each complementary region is a planar surface of Euler characteristic -1, is a *pants decomposition* of F and contains $N = 3g - 3 + s$ curves. Given a point $\Gamma \in T(F)$, there are associated hyperbolic lengths $\ell_i > 0$ of the Γ -geodesic representatives of the α_i ; these are completed to *Fenchel-Nielsen coordinates* on $T(F)$ by adjoining for each α_i one real *twisting parameter* θ_i which records the relative displacement of the hyperbolic structures across the geodesic curves, for $i = 1, \dots, N$.

If $\sigma \subseteq \mathcal{P}$, then there is a corresponding stratum T_σ added to $T(F)$ in $\bar{T}(F)$ as follows: the range of the Fenchel-Nielsen coordinates with respect to \mathcal{P} is extended to allow $\ell_i \geq 0$, for $\alpha_i \in \sigma$, where the twisting parameter θ_i is undefined for $\ell_i = 0$. Geometrically, one pinches each curve in σ to produce a nodal surface F_σ , and the added stratum T_σ is a copy of the products of the Teichmüller spaces of all the irreducible components of F_σ . In particular, $T_\emptyset \approx T(F)$ is the *null stratum*.

Definition 4.1. The *augmented Teichmüller space* of F is the space

$$\bar{T}(F) = T_\emptyset \cup \bigcup T_\sigma,$$

where the union is over all non-empty curve families σ in F , topologized so that a neighborhood of a point \bar{F} in T_σ , for σ a subset of a pants decomposition \mathcal{P} , consists of those points whose hyperbolic lengths are close to those of \bar{F} for each curve in \mathcal{P} and whose twisting parameters are close to those of \bar{F} for each curve in $\mathcal{P} - \sigma$.

A mapping class $\varphi \in MC(F)$ acts on $\bar{T}(F)$ by push-forward of metric as usual on the null stratum and induces an action on other strata $T_\sigma \rightarrow T_{\varphi(\sigma)}$ via permutation of curve families and push-forward of metric on irreducible components. In particular, the pure mapping class group of each irreducible component acts on its corresponding factor Teichmüller space in each stratum. For us, an important point [3, 23] is that the *Deligne-Mumford compactification* of Riemann's moduli space, denoted $\bar{\mathcal{M}}(F)$ is

$$\bar{\mathcal{M}}(F) = \bar{T}(F)/MC(F).$$

Definition 4.2. Associated with each stratum T_σ is its *stratum graph* $\mathcal{G}(\sigma)$ which has a vertex for each irreducible component of F_σ and an edge for each component of σ itself connecting the vertices corresponding to the irreducible components (which may coincide) on its two sides; for each puncture of the original surface F , we add yet another edge to $\mathcal{G}(\sigma)$ connecting a univalent vertex indicated by the special icon $*$ with the vertex corresponding to the irreducible component of F_σ containing it. See Figure 11 for an example.

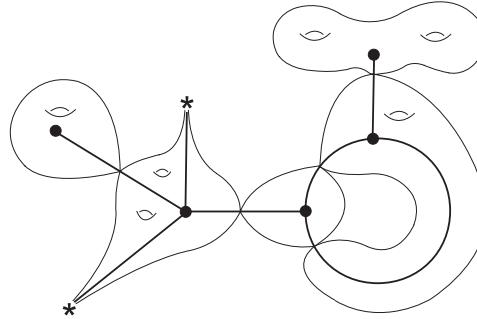


FIGURE 11. The stratum graph $\mathcal{G}(\sigma)$ associated with the nodal surface F_σ .

We will typically identify the edges of a stratum graph with $N \cup P$, where N is the set of nodes and P is the set of punctures of the nodal surface. By construction, a stratum graph is always connected and contains at least one $*$ -vertex. It follows that there is a path in the stratum graph from any element of $N \cup P$ to a $*$ -vertex.

Given a stratum $T_\sigma \subset \bar{T}(F)$ and an $\alpha_i \in \sigma$ with $\ell_i = 0$, we can increase ℓ_i , and in so doing move to the stratum $T_{\sigma - \{\alpha_i\}}$ via the resolution of the

node of F_σ corresponding to α_i , in which we replace the double point with α_i . On the level of stratum graphs, this resolution corresponds to collapsing an edge x of $\mathcal{G}(\sigma)$ to obtain $\mathcal{G}(\sigma - \{\alpha_i\})$, with the caveat that if an edge has its endpoints identified, we simply remove it from the graph.

4.2. Nests. Fix some stratum graph $\mathcal{G}(\sigma)$ with edges $N \cup P$. We shall consider functions

$$f : N \cup P \rightarrow \{0\} \cup \mathbb{N}$$

and refer to $f(x)$ as the *level* of $x \in N \cup P$. We say that f is *admissible* if for every $n \in N$ there is an adjacent $x \in N \cup P$ with $f(x) < f(n)$, where here x is *adjacent* to n if they share a common vertex endpoint.

Definition 4.3. A *nest* on a stratum graph \mathcal{G} is a function $f : N \cup P \rightarrow \{0\} \cup \mathbb{N}$ such that:

1. there is some $p \in P$ with $f(p) = 0$;
2. $f(n) > 0$, for all $n \in N$;
3. f is admissible;
4. f maps onto $\{0, 1, \dots, \max_{x \in N \cup P} f(x)\}$.

It follows in particular from the next result that every stratum graph admits a nest.

Lemma 4.4. *Given a stratum graph $\mathcal{G}(\sigma)$ and $x \in N \cup P$, let $f_\sigma(x)$ denote the least number of non-* vertices traversed along a path in $\mathcal{G}(\sigma)$ from x to a *-vertex. Then the function f_σ is a nest on $\mathcal{G}(\sigma)$.*

Proof. By definition, $f_\sigma(p) = 0$ for all $p \in P \neq \emptyset$, so condition 1 holds, and condition 2 is likewise clear. A path γ which minimizes the vertex count for the edge $n \in N$ must pass through one of its boundary vertices, and the next edge visited by γ must have level exactly one less than n . This proves both conditions 3 and 4. \square

Moreover, nests arise naturally from the space of filtered screens in the following manner: A stable path \bar{X}_t based on some i.c.d. E of F has its corresponding limiting filtered screen $\vec{\mathcal{E}} = \vec{\mathcal{E}}(\bar{X}_t) \subset \mathcal{FS}(F)$ whose boundary $\partial\vec{\mathcal{E}}$ is a curve family on F . We add to this curve family also the horocycles in F itself to get a collection of pairwise disjoint simple closed curves γ in F , each of which is uniquely represented as a closed edge path $e_0 - e_1 - \dots - e_k$ in the fatgraph $G(E)$ and for which we may define a length $\nu_t(\gamma) = \sum_{i=0}^k X_t(e_i)$ for some deprojectivization X_t of \bar{X}_t . Stability of \bar{X}_t implies that ratios $\nu_t(\gamma_1)/\nu_t(\gamma_2)$ are asymptotically well-defined for any two such cycles γ_1, γ_2 , and so there are well-defined comparability classes of these quantities. Insofar as the stratum graph

$$\mathcal{G}(\bar{X}_t) = \mathcal{G}(\partial\vec{\mathcal{E}}(\bar{X}_t))$$

obtained by pinching $\partial\vec{\mathcal{E}}$ has its edges naturally identified with these cycles, there is an induced function $f_{\bar{X}_t}(x)$ given by the number of strictly smaller such comparability classes for each edge x of $\mathcal{G}(\bar{X}_t)$.

Corollary 4.5. *The function $f_{\bar{X}_t}$ is a nest on $\mathcal{G}(\bar{X}_t)$.*

Proof. We must verify that the conditions of Definition 4.3 are satisfied. Note that the smallest comparability class of cycle in the definition of $f_{\bar{X}_t}$ has level zero by construction, and we may deprojectivize X_t so that its length is bounded away from zero.

There must therefore be some edge comprising this cycle with its simplicial coordinate likewise bounded, and this edge occurs in either one or two cycles corresponding to horocycles, each of which must also have level zero. This confirms condition 1. For condition 2, since $\partial \vec{\mathcal{E}}$ agrees with the boundary of a corresponding screen of divergent lambda lengths according to Corollary 3.22, the lambda length of each edge comprising a cycle in $\partial \vec{\mathcal{E}}$ diverges, hence its simplicial coordinate vanishes compared to level zero according to Lemma 3.15 as required. An edge of $\mathcal{G}(\bar{X}_t)$ corresponding to a node is least level at one of its endpoints if and only if the associated puncture in the punctured fatgraph $\bar{G}(p(\bar{X}_t))$ is decorated; since at most one in a pair of paired punctures can be decorated, condition 3 holds. Since condition 4 holds by construction, the proof is complete. \square

We will use nests to decorate augmented Teichmüller space, but before doing so, we require some notation and properties concerning nests on stratum graphs \mathcal{G} .

4.3. Modifying nests and stratum graphs. Given a nest f on \mathcal{G} , we may always decrease non-zero levels of edges, which in the context of Corollary 4.5 corresponds to reducing the rate of vanishing of sums of simplicial coordinates along horocycles or cycles in stable paths. However, the resulting function may no longer be a nest on \mathcal{G} , in particular since condition 3 of Definition 4.3 may fail for certain edges, which again in the context of Corollary 4.5 and its preceding discussion means pinch curves represented by such edges will no longer be in the boundary of the limiting filtered screen. We therefore may have to modify the stratum graph by collapsing edges, which corresponds to resolving nodes in the nodal surface.

Here we specify precisely which edges to collapse if we are decreasing by one the levels of a subset of edges; this will also serve to introduce definitions and notation which will be useful in the sequel.

Definition 4.6. Given a stratum graph \mathcal{G} and a function $f : N \cup P \rightarrow \{0\} \cup \mathbb{N}$, define

$$L_f^k = \{x \in N \cup P : f(x) = k\}$$

These are the edges of level k for f ; we will refer to the L_f^k as *comparability classes* or *levels* of f . If f satisfies all the conditions of a nest in Definition 4.3 except for condition 4, we define $[f](x)$ to be the number of strictly smaller nonempty comparability classes for each $x \in N \cup P$; it is easy to see that $[f]$ is a nest canonically defined by f in this case.

Definition 4.7. Given a nest f on \mathcal{G} and $M \subset ((N \cup P) - L_f^0)$, define

$$f_M(x) = \begin{cases} 2f(x), & \text{if } x \in (N \cup P) - M; \\ 2f(x) - 1, & \text{if } x \in M. \end{cases}$$

In effect, f_M changes f by inserting additional levels which isolate edges of M in their own new comparability classes. It is evident that $\lfloor f_M \rfloor$ is a nest on \mathcal{G} .

As we proceed, we will adopt the convention that whenever we modify non-zero levels of a nest on a stratum graph, we will always conclude the modification by applying the $\lfloor \cdot \rfloor$ operator to the resulting function; however, typically we will omit the $\lfloor \cdot \rfloor$ notation. To prove that the resulting function is a nest, it will therefore suffice to verify conditions 1-3 of Definition 4.3.

Definition 4.8. Given a nest f on \mathcal{G} and $M \subset ((N \cup P) - L_f^0)$, define

$$f^M(x) = \begin{cases} f(x), & \text{if } x \in (N \cup P) - M; \\ f(x) - 1, & \text{if } x \in M, \end{cases}$$

which simply decreases f -levels of M by one. Define $C(M)$ to be all $n \in N$ for which $f^M(n) \leq f^M(x)$ for all edges x adjacent to n , that is, all $n \in N$ that cause f^M to be inadmissible; it is clear that $C(M) \subset (M \cap N)$.

Before presenting a lemma which characterizes edges in $C(M)$, we introduce some notation. If $x \in N \cup P$, let $V(x)$ be the set of non-* vertex endpoints of x . Given a non-* vertex v , let $\text{Star}(v)$ be those edges with v as an endpoint. For a nest f and a non-* vertex v , define $\text{Min}_f(v)$ to be the edges of least f -level in $\text{Star}(v)$.

Lemma 4.9. Given a nest f on \mathcal{G} and $M \subset ((N \cup P) - L_f^0)$, we have $n \in C(M)$ if and only if one of the following holds:

1. For each $v \in V(n)$ there exists $x \in \text{Min}_f(v) - M$ such that $f(x) + 1 = f(n) \leq f(m)$ for all $m \in M \cap \text{Star}(v)$.
2. There exists $v_1 \in V(n)$ and $x \in \text{Min}_f(v_1) - M$ such that $f(x) + 1 = f(n) \leq f(m)$ for all $m \in M \cap \text{Star}(v_1)$, and there exists $v_2 \in V(n)$ such that $n \in \text{Min}_f(v_2)$.

Proof. The proof is essentially by the definition of $C(M)$. Decreasing levels of M by one in passing from f to f^M will result in $f^M(n) \leq f^M(x)$ for all x adjacent to $n \in M$ if and only if one of the above two conditions holds depending on whether $n \in \text{Min}_f(v)$ for some $v \in V(n)$. \square

Given a nest f on \mathcal{G} and $M \subset ((N \cup P) - L_f^0)$, collapse all $x \in C(M)$ to obtain a new stratum graph $\mathcal{G}_{C(M)}$; the function f^M restricts to $\mathcal{G}_{C(M)}$, and we have the following:

Lemma 4.10. *Given a nest f on \mathcal{G} and $M \subset ((N \cup P) - L_f^0)$, the function $f^M|_{\mathcal{G}_{C(M)}}$ is a nest on $\mathcal{G}_{C(M)}$.*

Proof. We verify conditions 1,2, and 3 for f^M on $\mathcal{G}_{C(M)}$, and condition 1 holds by construction. If $f(n) = 1$ for $n \in M \cap N$, then $n \in C(M)$ and is therefore collapsed; thus, $f^M(n) > 0$ for all $n \in (N - C(M))$, and condition 2 holds.

To verify condition 3, let $n \in \mathcal{G}$ be an edge not in $C(M)$. Since the original f on \mathcal{G} is a nest, there is an edge x adjacent to n with $f(x) < f(n)$; call their common vertex v_1 . Consider an edge path $n = x_0 - x_1 - \dots - x_k \in P$, defined recursively by $x_i \in \text{Min}_f(v_i)$ for all $i > 0$, where v_i is the common vertex of x_{i-1} and x_i , and if there exists $e_i \in (\text{Min}_f(v_i) \cap M)$, then we let x_i be such an e_i . By definition, $f(x_i) > f(x_{i+1})$ for all i . If we now consider f^M on this same edge-path in \mathcal{G} , there may be pairs of consecutive edges $x_i - x_{i+1}$ for which $f^M(x_i) = f^M(x_{i+1})$. In this case, it is evident that x_i satisfies condition 2 of Lemma 4.9 for f , thus lies in $C(M)$ and is hence collapsed in passing to $\mathcal{G}_{C(M)}$. Thus, the induced edge-path $n = x_0 - x_{i_1} - \dots - x_{i_j} \in P$ in $\mathcal{G}_{C(M)}$ satisfies $f^M(x_\ell) > f^M(x_{\ell+1})$, for each ℓ , and condition 3 holds for n . \square

If $C(M) = \emptyset$, we will refer to the change from f to f^M as a *permissible decrease* of f -level for M , since by the above lemma f^M is a nest on $\mathcal{G} = \mathcal{G}_\emptyset$. In the special case that $M = L_f^{k+1}$ for some $k \geq 0$, with $C(M) = \emptyset$, we will refer to this as a *permissible coalescing of adjacent levels*, as the result is to merge L_f^{k+1} with L_f^k into one level. These modifications will play a special role corresponding to codimension-one faces in decorated augmented Teichmüller space.

Definition 4.11. If $C(M) = M$, we will say that M is a *collapsable set of edges* for f , and if $M = C(M) = \{n\}$, we will say that the edge n is *collapsible* for f . In the special case that $M = L_f^k$ for some k , with $M = C(M)$, we will refer to this as a *collapsable level of edges*, which will also be key to defining decorated augmented Teichmüller space.

Given a collapsible set of edges M for f , we may always induce a nest on \mathcal{G}_M via the sequence of nests given by

$$(3) \quad f \text{ on } \mathcal{G} \rightarrow f_M \text{ on } \mathcal{G} \rightarrow (f_M)^M \text{ on } \mathcal{G}_M$$

We will call such a sequence a *collapsing sequence*.

We observe the following lemma for the case where $M = \{n\}$:

Corollary 4.12. *For a fixed stratum graph \mathcal{G} and nest f , an edge $n \in N$ is collapsible for f if and only if n is collapsible for $f_{\{n\}}$.*

Proof. The result follows immediately from Lemma 4.9. \square

We conclude with two further corollaries of Lemma 4.9.

Corollary 4.13. *Given a nest f on \mathcal{G} and sets M_1, M_2 of edges of \mathcal{G} , we have the inclusion $C(M_1 \cup M_2) \subset C(M_1) \cup C(M_2)$.*

Proof. Any $n \in C(M_1 \cup M_2) \subset M_1 \cup M_2$ must satisfy either condition 1 or 2 of Lemma 4.9. The same condition must therefore hold for either $n \in M_1$ or $n \in M_2$, and thus $n \in C(M_1)$ or $n \in C(M_2)$. \square

Corollary 4.14. *Given a nest f on \mathcal{G} and any $M \subset ((N \cup P) - L_f^0)$, there exists a maximal subset $\mathring{M} \subset M$ for which $C(\mathring{M}) = \emptyset$.*

4.4. Decorated augmented Teichmüller space. Now we decorate augmented Teichmüller space using nests.

Definition 4.15. Given a nest f on a stratum graph $\mathcal{G}(\sigma)$ with edges $N \cup P$ and $\max_{x \in N \cup P} f(x) = k$, the *cell associated to f* is the product of simplices

$$\mathcal{C}(f) = P\mathbb{R}_{>0}^{L_f^0} \times P\mathbb{R}_{>0}^{L_f^1} \times \cdots \times P\mathbb{R}_{>0}^{L_f^k}$$

In the context of Corollary 4.5, the coordinates in a particular factor simplex may be interpreted as the projectivized sums of simplicial coordinates along cycles parallel to short curves or horocycles within the same comparability class.

Given a stratum graph $\mathcal{G}(\sigma)$, cells $\mathcal{C}(f)$ associated to nests f on $\mathcal{G}(\sigma)$ form a simplicial complex, defined below, which we will see serve as fibers over strata in decorated augmented Teichmüller space.

Definition 4.16. Given a fixed stratum graph $\mathcal{G}(\sigma)$, define $\mathcal{D}(\sigma)$ as

$$\mathcal{D}(\sigma) = \left[\bigsqcup_{\text{nests } f \text{ on } \mathcal{G}(\sigma)} \mathcal{C}(f) \right] / \sim,$$

where the face relation \sim is generated by permissible coalescing of adjacent levels, that is, $\mathcal{C}(f)$ is a codimension-one face of $\mathcal{C}(f')$ if $f' = f^M$ for $M = L_f^{k+1}$ for some $k \geq 0$.

In terms of coordinates, one moves from $\mathcal{C}(f')$ to $\mathcal{C}(f)$ by allowing the coordinates associated with $L_f^{k+1} \subset L_{f'}^k$ in $\mathcal{C}(f')$ to go to zero all at a comparable rate; the new barycentric coordinates associated with L_f^{k+1} in $\mathcal{C}(f)$ are then projectivized limits of the vanishing coordinates in that path, and the new barycentric coordinates associated with L_f^k in $\mathcal{C}(f)$ are the projectivization of the bounded coordinates in that path.

Definition 4.17. The *decorated augmented Teichmüller space* is the subspace of

$$\bar{T}(F) \times \left[\bigsqcup_{\text{curve families } \sigma} \mathcal{D}(\sigma) \right] / \sim$$

given by

$$\hat{T}(F) = \{(\bar{F}, \bar{x}) : \bar{F} \in T_\sigma \text{ and } \bar{x} \in \mathcal{C}(f) \text{ for a nest } f \text{ on } \mathcal{G}(\sigma)\},$$

where the face relation \sim across the $\mathcal{D}(\sigma)$ is generated by the following: For every nest f on \mathcal{G} with collapsible level $M = L_f^k$, we identify the non- M coordinates of $\mathcal{C}(f)$ on \mathcal{G} with the associated coordinates of $\mathcal{C}(f^M|_{\mathcal{G}_M})$ on \mathcal{G}_M .

Remark 4.18. In the specific case where $M = \{n\}$ for $n \in N$, suppose $\mathcal{G}(\sigma')$ is the stratum graph obtained from $\mathcal{G}(\sigma)$ by collapsing n , and let f be a nest on $\mathcal{G}(\sigma)$. By Definition 4.17, $\mathcal{C}(f)$ shares a common boundary face with $\mathcal{D}(\sigma')$, namely $\mathcal{C}(f_{\{n\}})$, if and only if n is collapsible for $f_{\{n\}}$, which holds by Corollary 4.12 if and only if n is collapsible for f . If this is the case, we will say that $\mathcal{C}(f)$ is *adjacent to* $\mathcal{D}(\sigma')$.

There is a natural continuous projection from $\hat{T}(F) \rightarrow \bar{T}(F)$, where the fiber over a point $\bar{F} \in T_\sigma$ is $\mathcal{D}(\sigma)$. We can extend the action of $MC(F)$ on $\bar{T}(F)$ to $\hat{T}(F)$ in the natural way, namely, by push-forward not only of metric but also nest values on the edges of stratum graphs.

4.5. Homotopy equivalence with augmented Teichmüller space. We now show that the projection $\hat{T}(F) \rightarrow \bar{T}(F)$ is an equivariant homotopy equivalence and begin with preliminary lemmas. For a fixed stratum graph $\mathcal{G}(\sigma)$, recall the nest f_σ defined in Lemma 4.4. Given another nest f , define $m_f = \min\{k : L_f^k \neq L_{f_\sigma}^k\}$, that is, the least level at which f disagrees with f_σ .

Lemma 4.19. $L_f^{m_f} \subset L_{f_\sigma}^{m_f}$.

Proof. The lemma is true for $m_f = 0$ since by definition $L_f^0 \subset P = L_{f_\sigma}^0$. We therefore assume that $m_f > 0$. For the induction, we have $L_f^k = L_{f_\sigma}^k$ for all $k < m_f$ and so consider a nodal edge $n \in [(N \cup P) - \cup_{i=0}^{m_f-1} L_{f_\sigma}^i]$ with $f(n) = m_f$. By condition 3 of Definition 4.3 for a nest, there is an edge e adjacent to n for which $f(e) < m_f$, implying that $f_\sigma(e) < m_f$. By the definitions of f_σ and m_f , we therefore must have $f_\sigma(n) = m_f$. \square

With this lemma in mind, define $M(f) = L_{f_\sigma}^{m_f} - L_f^{m_f}$, that is, all edges of level m_f in f_σ that are not of level m_f in f . Consider $f^{M(f)}$ and observe that $f^{M(f)}(x) \leq f(x)$ for all $x \in N \cup P$. Note also that Lemma 4.19 shows that this $.^M$ -operator moves a nest f progressively toward f_σ . Furthermore, we have the following:

Lemma 4.20. *If f is a nest on any stratum graph $\mathcal{G}(\sigma)$, then $f^{M(f)}$ is also a nest. Moreover, beginning with $f = f_0$, there is a finite sequence of nests f_0, \dots, f_s , such that $f_s = f_\sigma$, and $f_{i+1} = (f_i)^{M(f_i)}$ for all i . In particular, we have $f_\sigma(x) \leq f(x)$ for all $x \in N \cup P$.*

Proof. To show that $f^{M(f)}$ is a nest, it suffices to show that $C(M(f)) = \emptyset$. To this end, note that if $n \in M(f)$, then $f_\sigma(n) = m_f$. By the definition of f_σ , there then must be an x adjacent to n for which $f_\sigma(x) = m_f - 1 = f(x)$ by the definition of m_f . Thus, by the definition of $M(f)$, we have $f(n) > f_\sigma(n) = f(x) + 1$, and by Lemma 4.9, we conclude that $n \notin C(M(f))$.

We finish the proof of the lemma by simply observing that we can begin with f and repeatedly apply \cdot^M until we reach an f_s such that $L_{f_s}^k = L_{f_\sigma}^k$ for all k using Lemma 4.19. \square

Given a nest f on $\mathcal{G}(\sigma)$, we now define $\bar{M}(f) = \{x \in N \cup P : f(x) > f_\sigma(x)\}$, and let $\mathring{M}(f) \subset \bar{M}(f)$ be the maximal subset of $\bar{M}(f)$ for which $C(\mathring{M}(f)) = \emptyset$, coming from Corollary 4.14. A corollary of Lemma 4.20 is then the following:

Corollary 4.21. *For any nest f on a stratum graph $\mathcal{G}(\sigma)$, there is a finite sequence of nests $f = f_0, \dots, f_s = f_\sigma$ such that $f_{i+1} = (f_i)^{\mathring{M}(f_i)}$ for all i .*

Proof. As long as $f_i \neq f_\sigma$, Lemma 4.20 shows that $(L_{f_\sigma}^{m_f} - L_f^{m_f}) \subset \bar{M}(f)$ satisfies $C(L_{f_\sigma}^{m_f} - L_f^{m_f}) = \emptyset$. Thus, $\mathring{M}(f)$ will be non-empty, and the $\cdot^{\mathring{M}}$ -operator will move f_i progressively toward the minimal f_σ . \square

Lemma 4.22. *Given a nest f on $\mathcal{G}(\sigma)$, if $n \in N$ is collapsable for f , then n is also collapsable for $f^{\mathring{M}(f)}$.*

Proof. We first claim that if $n \notin \bar{M}(f)$, meaning that $f(n) = f_\sigma(n)$, then n is necessarily collapsable both for $g_1 = f$ and $g_2 = f^{\mathring{M}(f)}$. To see this, denote the vertices in $V(n)$ by v_1, v_2 . If $n \in \text{Min}_{g_i}(v_2)$, then there exists $x \in \text{Min}_{g_i}(v_1)$ with $g_i(x) < g_i(n) = f_\sigma(n)$. By the definition of f_σ and Lemma 4.20, we must have $g_i(x) \geq f_\sigma(x) \geq f_\sigma(n) - 1$, and therefore $g_i(x) = f_\sigma(x) = g_i(n) - 1$, and we see that n satisfies 2 of Lemma 4.9. On the other hand, if $n \notin \text{Min}_{g_i}(v_j)$ for either $j = 1$ or 2 , then for both v_j , there exists such an x_j , and n satisfies 1 of Lemma 4.9 and is again collapsable.

If $n \in \mathring{M}(f)$ is collapsable for f , then n will clearly be collapsable for $f^{\mathring{M}(f)}$. The remaining case is then $n \in (\bar{M}(f) - \mathring{M}(f))$. If n satisfies 1 or 2 of Lemma 4.9 for the nest f , then the associated $x \in \text{Min}_f(v)$ cannot be in $\mathring{M}(f)$, for otherwise, n could have been added to $\mathring{M}(f)$ preserving $C(\mathring{M}(f)) = \emptyset$ contradicting the fact that $\mathring{M}(f)$ is maximal. As a result, n satisfies 1 or 2 of Lemma 4.9 for $f^{\mathring{M}(f)}$ and is thus collapsable. \square

We conclude with the following proposition, our goal for this section:

Proposition 4.23. *The projection $\hat{T}(F) \rightarrow \bar{T}(F)$ is an equivariant homotopy equivalence.*

Proof. The proof is an application of the algorithm using the $\cdot^{\mathring{M}}$ -operator in Corollary 4.21. Specifically, given any point $(\bar{F}, \bar{x}) \in \hat{T}(F)$, where $\bar{F} \in T_\sigma$ and $\bar{x} \in \mathcal{C}(f)$ for a nest f on $\mathcal{G}(\sigma)$, we can apply the $\cdot^{\mathring{M}}$ -operator to the nest

f while maintaining (deprojectivized) simplicial coordinates determined by \bar{x} for edges of $\mathcal{G}(\sigma)$. We observe that in doing so, when we decrease the levels of $\mathring{M}(f)$ by one, this is in fact accomplished by homotopy within the fiber $\mathcal{D}(\sigma)$ since $C(\mathring{M}(f)) = \emptyset$, and thus takes the point $\bar{x} \in \mathcal{C}(f)$ to a point $\bar{y} \in \mathcal{C}(f^{\mathring{M}(f)})$ in $\mathcal{D}(\sigma)$, where \bar{y} is obtained by projectivizing (within the comparability classes of $f^{\mathring{M}(f)}$) the coordinates on $\mathcal{G}(\sigma)$ determined by \bar{x} . We call this flow generated by the $\cdot^{\mathring{M}}$ -operator the $\cdot^{\mathring{M}}$ -flow.

The $\cdot^{\mathring{M}}$ -flow is continuous within a particular $\mathcal{D}(\sigma)$, for suppose that in $\mathcal{D}(\sigma)$ we have that $\mathcal{C}(f)$ is a face of $\mathcal{C}(f')$, so that f' is obtained by permissible coalescing of two adjacent levels of f , say, L_f^k and L_f^{k+1} . Since clearly $L_f^{\geq k+1} = \cup_{i=k+1}^n L_f^i \subset \mathring{M}(f)$, the $\cdot^{\mathring{M}}$ -flow thus achieves the decreasing of level by one for $L_f^{\geq k+1}$.

Moreover, we claim that the $\cdot^{\mathring{M}}$ -flow is continuous across adjacent strata. Indeed, if $\mathcal{C}(f)$ in $\mathcal{D}(\sigma)$ is adjacent to $\mathcal{D}(\sigma')$, where $\mathcal{G}(\sigma')$ is obtained from $\mathcal{G}(\sigma)$ by collapsing an edge n , then n must be collapsible for f by Remark 4.18. By Lemma 4.22, n is also collapsible for $f^{\mathring{M}(f)}$, and therefore $\mathcal{C}(f^{\mathring{M}(f)})$ is still adjacent to $\mathcal{D}(\sigma')$.

We may therefore perform a global homotopy generated by the $\cdot^{\mathring{M}}$ -flow, with image in $\mathcal{C}(f_\sigma)$ for every fiber $\mathcal{D}(\sigma)$ by Corollary 4.21. This flow is $MC(F)$ -equivariant, and since each $\mathcal{C}(f_\sigma)$ is contractible, the result follows. \square

5. CELL DECOMPOSITION AND EQUIVARIANT HOMOTOPY EQUIVALENCE

In this section, we finally prove our main theorems, establishing an $MC(F)$ -invariant cell decomposition for $\mathcal{PG}(F)$, as well as an $MC(F)$ -equivariant homotopy equivalence from $\mathcal{PG}(F)$ to $\bar{T}(F)$. To do so, we first connect $\hat{T}(F)$ to both $\mathcal{PG}(F)$ and $\mathcal{FS}(F)$ via maps to and from those respective spaces.

5.1. Maps to and from decorated augmented Teichmüller space.

Definition 5.1. Define a map $\chi : \hat{T}(F) \rightarrow \mathcal{PG}(F)$ as follows: Consider a point $(\bar{F}, \bar{x}) \in \hat{T}(F)$, which corresponds to a (possibly) nodal surface $\bar{F} \in T_\sigma$, together with a point $\bar{x} \in \mathcal{C}(f)$ corresponding to a nest f on $\mathcal{G}(\sigma)$. Each irreducible component F_i of \bar{F} is represented by a non-* vertex v_i in $\mathcal{G}(\sigma)$, and each such vertex v_i has edges in $\text{Min}_f(v_i)$. We determine $\chi(\bar{F}, \bar{x})$ by applying the convex hull construction to each irreducible component F_i of \bar{F} using only the decorations on its (possibly nodal) punctures associated to edges in $\text{Min}_f(v_i)$ finally projectivizing simplicial coordinates in the resulting punctured fatgraph. The result is a labeled punctured fatgraph with partial pairing, which we denote by $\bar{G} = \chi(\bar{F}, \bar{x})$, where the pairing is given by the nodal pairing in \bar{F} .

Lemma 5.2. *Given $(\bar{F}, \bar{x}) \in \hat{T}(F)$, $\chi(\bar{F}, \bar{x}) = \bar{G}$ indeed lies in $\mathcal{PG}(F)$, and thus χ is well-defined.*

Proof. We need to construct a fatgraph G for F and an associated filtered screen $\vec{\mathcal{E}}$ on G , along with a point $p \in \mathcal{C}(\vec{\mathcal{E}})$, such that $\pi(p) = \chi(\bar{F}, \bar{x})$.

To this end, we take the punctured fatgraph with pairing \bar{G} , and resolve all punctured vertices and pairings (non-uniquely) using the operations in Figure 12, to obtain a fatgraph G representing F .

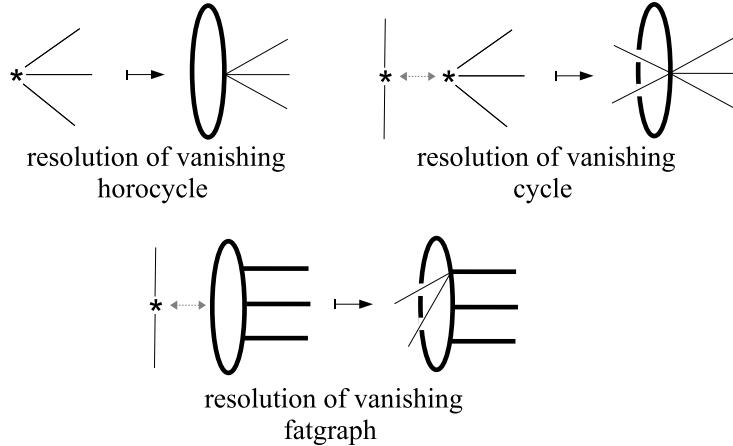


FIGURE 12. Chosen resolutions of pairings and punctured vertices of \bar{G} to obtain a fatgraph G .

We note here that the admissible condition on the nest associated to (\bar{F}, \bar{x}) precludes paired decorated punctures since nodal edges cannot be least level for both endpoints. Thus, the operations in Figure 12 are exhaustive.

This chosen resolution produces a fatgraph G for F . The level structure for the filtered screen on G is given as follows: For each edge e in G which is also in \bar{G} , the level of e will be the f -level of edges in $\text{Min}_f(v_i)$, where v_i is the non-* vertex representing the irreducible component F_i containing $e \in \bar{G}$. Furthermore, for edges of G in horocycles or cycles resulting from the resolution of \bar{G} , we assign the f -level associated to the corresponding edge in $\mathcal{G}(\sigma)$. The claim is that this level structure yields a filtered screen, and if the maximum level is n , it suffices to show that $L^{\geq k}$ is recurrent for all $k = 1, \dots, n$. To see this, fix k and consider a component of L^k ; if this component is a horocycle, cycle, or fatgraph, then it is recurrent. Otherwise, the edges in the component of L^k appear in \bar{G} as a punctured fatgraph with punctured vertices, corresponding to an irreducible component F_i . However, in G these punctured vertices no longer appear, and edges incident on such punctured vertices of \bar{G} are now incident on a vertex of G in either a vanishing horocycle, cycle, or fatgraph, all of which are of greater level, and hence in $L^{\geq k}$. Thus, $L^{\geq k}$ is recurrent, and the level structure yields a filtered screen $\vec{\mathcal{E}}$.

Finally, we need to determine correct coordinates for a point $p \in \mathcal{C}(\vec{\mathcal{E}})$ so that $\pi(p) = \chi(\bar{F}, \bar{x})$. The (possibly deprojectivized) coordinates for any

horocycle or cycle in G resulting from a resolution can simply be assigned arbitrarily. For all other edges, the (possibly deprojectivized) coordinates are identical to those in $\chi(\bar{F}, \bar{x})$; now projectivize in comparability classes in the level structure of $\vec{\mathcal{E}}$ to obtain $p \in \mathcal{C}(\vec{\mathcal{E}})$ for which $\pi(p) = \chi(\bar{F}, \bar{x})$. \square

We next show that χ is in fact surjective, i.e., every labeled punctured fatgraph with partial pairing $\bar{G} \in \mathcal{PG}(F)$ can be obtained by decorating only those punctures in each irreducible component corresponding to edges of least level for a nest on the associated stratum graph. In so doing, we will introduce the notion of a *partially oriented stratum graph*, which will also prove useful in describing the cell decomposition for $\mathcal{PG}(F)$.

Definition 5.3. To every point $\bar{G} \in \mathcal{PG}(F)$ we associate a unique stratum graph $\mathcal{G}_{\bar{G}}$, which has a non-* vertex for each component punctured fatgraph and an edge for each pair of paired punctures or puncture paired with a boundary component connecting the vertices corresponding to the punctured fatgraphs (which may coincide) on its two sides; for each unpaired puncture, we add yet another edge connecting a univalent *-vertex to the vertex of the associated punctured fatgraph.

Definition 5.4. For every stratum graph $\mathcal{G}_{\bar{G}}$, we uniquely orient a subset of its edges to obtain a *partially oriented stratum graph* $\vec{\mathcal{G}}_{\bar{G}}$, as follows: for edges corresponding to a pairing of a decorated with an undecorated puncture, we orient from the decorated towards the undecorated puncture; for edges corresponding to unpaired decorated punctures, we orient towards the unpaired *-vertex; all other edges are left unoriented. See Figure 13 for an example. In general, a *partial orientation* on a stratum graph will be an assignment of orientations to a subset of its edges.

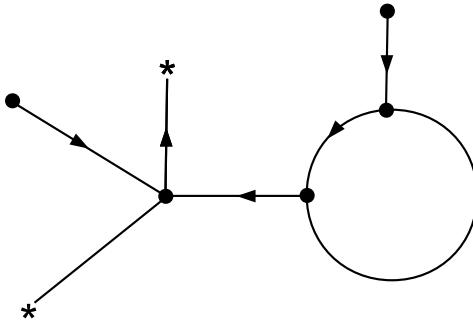


FIGURE 13. An example of a partially oriented stratum graph $\vec{\mathcal{G}}_{\bar{G}}$.

In other words, given $\mathcal{G}_{\bar{G}}$, we orient edges in the direction of decreasing level in filtered screens in the preimage $\pi^{-1}(\bar{G})$; as a result, there do not exist oriented cycles in $\vec{\mathcal{G}}_{\bar{G}}$. For each vertex v , we therefore define $d(v)$ to be the maximal count of non-* vertices traversed by such an oriented path

starting from v without counting v itself; we will call such a path realizing this maximal count a *maximal path* for v .

Definition 5.5. Given a partially oriented stratum graph $\vec{\mathcal{G}}_{\bar{G}}$ associated with a point $\bar{G} \in \mathcal{PG}(F)$, define $f_{\bar{G}} : N \cup P \rightarrow \{0\} \cup \mathbb{N}$ as follows: If $x \in N \cup P$ is an oriented edge, define $f_{\bar{G}}(x) = d(v)$ where v is the initial vertex for the oriented x . If $x \in N \cup P$ is unoriented, define $f_{\bar{G}}(x)$ to be one greater than $\max\{d(v) | v \in V(x)\}$.

Lemma 5.6. *The function $f_{\bar{G}}$ is a nest on $\mathcal{G}_{\bar{G}}$.*

Proof. First, choose $p \in \mathcal{C}(\vec{\mathcal{E}}) \subset \mathcal{FS}(F)$ such that $\pi(p) = \bar{G}$ and observe that in $\mathcal{G}_{\bar{G}}$, there is a vertex v associated to a punctured fatgraph component in \bar{G} corresponding to level zero edges in the screen $\vec{\mathcal{E}}$ for which $p \in \mathcal{C}(\vec{\mathcal{E}})$. This punctured fatgraph component will necessarily have a decorated puncture and thus an oriented edge with v as its initial point; by definition in Construction 3.26, all of its decorated punctures are unpaired. Thus, $d(v) = 0$, and condition 1 of Definition 4.3 is satisfied. Condition 2 is satisfied since any oriented $n \in N$ will have initial vertex v with $d(v) > 0$. To see that condition 3 holds, first observe that for an unoriented $n \in N$, there are by definition adjacent x for which $f_{\bar{G}}(x) < f_{\bar{G}}(n)$. If n is oriented, assume it is in $\text{Min}_{f_{\bar{G}}}(v)$ for some $v \in V(n)$ and call the opposite vertex v_0 . Consider directed paths beginning at v_0 , and observe that a maximal path beginning at v_0 must have length smaller than the maximal path beginning at v . Condition 3 therefore holds.

Finally, to confirm condition 4 of Definition 4.3, consider an oriented edge of level k . Its initial point is a vertex v with $d(v) = k$. Consider the maximal path for v , which must have length k . Along that path, the maximal length of a path for the next vertex after v must be $k - 1$. Thus, there must be an edge of level $k - 1$ in $f_{\bar{G}}$. A similar statement holds by definition for unoriented edges, and so condition 4 holds. \square

Lemma 5.7. *The map $\chi : \hat{T}(F) \rightarrow \mathcal{PG}(F)$ is a surjection.*

Proof. Given a point $\bar{G} \in \mathcal{PG}(F)$, each punctured fatgraph component determines a point in projectivized partially decorated Teichmüller space for the topological type of that component F_i , and hence an underlying hyperbolic structure on F_i , i.e., a point $\bar{F}(\bar{G}) \in T_\sigma \subset \bar{T}(F)$. Moreover, the nest $f_{\bar{G}}$ on $\mathcal{G}_{\bar{G}} = \mathcal{G}(\sigma)$ determines a cell $\mathcal{C}(f_{\bar{G}})$ in the fiber over T_σ , and decorated horocycles on each irreducible component of $\bar{G} \in \mathcal{PG}(F)$ give coordinates for the associated edges in the stratum graph $\mathcal{G}_{\bar{G}}$. Any other edges in the stratum graph $\mathcal{G}_{\bar{G}}$ may be arbitrarily assigned positive numbers, and then projectivizing within comparability classes of the same level yields coordinates in the product simplex associated to $\mathcal{C}(f_{\bar{G}})$. This is the required point $(\bar{F}, \bar{x}) \in \hat{T}(F)$ with $\chi(\bar{F}, \bar{x}) = \bar{G}$, so χ is indeed a surjection. \square

Remark 5.8. Since the nest $f_{\bar{G}}$ only depends on the underlying partially oriented stratum graph $\vec{\mathcal{G}}$, we will write simply $f_{\vec{\mathcal{G}}}$ for the same function.

Before proceeding to the map from $\mathcal{FS}(F)$ to $\hat{T}(F)$, we first more closely examine nests in $\chi^{-1}(\bar{G})$ and their relationship to $\vec{\mathcal{G}}_{\bar{G}}$. We also establish criteria which characterize the partially oriented stratum graphs $\vec{\mathcal{G}}_{\bar{G}}$.

Definition 5.9. Given a partially oriented stratum graph $\vec{\mathcal{G}}$, we say that a nest f on the underlying stratum graph \mathcal{G} is *compatible with* $\vec{\mathcal{G}}$ if for every vertex $v \in \mathcal{G}$, the set of oriented edges with initial point at v is precisely $\text{Min}_f(v)$.

Lemma 5.10. If $\chi(\bar{F}, \bar{x}) = \bar{G}$, then $\bar{x} \in \mathcal{C}(f)$ for some nest f compatible with $\vec{\mathcal{G}}_{\bar{G}}$. Conversely, if f is a nest compatible with $\vec{\mathcal{G}}_{\bar{G}}$, then there exists an $\bar{x} \in \mathcal{C}(f)$ and $\bar{F} \in T_\sigma$ such that $\chi(\bar{F}, \bar{x}) = \bar{G}$.

Proof. The first statement follows directly from the definition of the map χ and the definition of a nest being compatible with $\vec{\mathcal{G}}_{\bar{G}}$. The second statement again follows from these definitions with the added observation that we may choose appropriate coordinates for $\bar{x} \in \mathcal{C}(f)$ so that \bar{G} results from the convex hull construction applied to an appropriately chosen $\bar{F} \in T_\sigma$ using those coordinates as decorations on punctures of least level. \square

Definition 5.11. A partial orientation on a stratum graph \mathcal{G} is said to be *realizable* if the resulting partially oriented stratum graph $\vec{\mathcal{G}}$ supports a compatible nest, and we say that such a $\vec{\mathcal{G}}$ is a *realizable partially oriented stratum graph*.

The following lemma characterizes realizable orientations:

Lemma 5.12. A partial orientation on a stratum graph \mathcal{G} is realizable if and only if the following conditions are satisfied:

- i. Every non-* vertex is an initial point for an oriented edge.
- ii. There exists a non-* vertex v such that all oriented edges with initial point v terminate in a *-vertex.
- iii. There are no oriented cycles.

Proof. If $\vec{\mathcal{G}}$ supports a compatible nest f , then every vertex v has a non-empty set $\text{Min}_f(v)$, and thus i holds. Furthermore since f is a nest, there is at least one puncture of level zero, which will be represented by an oriented edge whose initial point satisfies ii. Finally, iii is clear since such an oriented cycle would result in the violation of condition 3 of a nest in Definition 4.3.

Conversely, given a $\vec{\mathcal{G}}$ satisfying i-iii, the function $f_{\vec{\mathcal{G}}}$ from Remark 5.8 yields a compatible nest. Specifically, this function is well-defined by iii, has at least one puncture of level zero by ii, and satisfies conditions 2,3, and 4 of a nest by reasoning identical to that in Lemma 5.6. \square

Our next lemma shows that the nest $f_{\vec{\mathcal{G}}}$ is in fact minimal for all nests compatible with a realizable partially oriented stratum graph $\vec{\mathcal{G}}$, a property which will be essential for establishing the homotopy equivalence of $\mathcal{PG}(F)$ with $\hat{T}(F)$:

Lemma 5.13. *If $\vec{\mathcal{G}}$ is a realizable partially oriented stratum graph and f is a nest compatible with $\vec{\mathcal{G}}$, then $f_{\vec{\mathcal{G}}}(x) \leq f(x)$ for all $x \in N \cup P$.*

Proof. For any non-* vertex v and $x \in \text{Min}_f(v)$, it must be the case that $f(x)$ is at least as large as the length of a maximal path beginning at v since f is a nest compatible with $\vec{\mathcal{G}}$. The result follows directly from this observation. \square

To conclude, we introduce a map $\psi : \mathcal{FS}(F) \rightarrow \hat{T}(F)$ defined as follows. Given a point $p \in \mathcal{C}(\vec{\mathcal{E}}) \subset \mathcal{FS}(F)$, we first define $\bar{F}(p)$ to be the underlying nodal hyperbolic surface determined by $\pi(p) = \bar{G}$ as in the proof of Lemma 5.7. Furthermore, if $x \in N \cup P$ in the stratum graph associated with $\pi(p)$, and if x is associated with a decorated (possibly nodal) horocycle, we assign to x the level of the corresponding horocycle (or cycle) in the filtered screen as well as a coordinate equal to the sum of the simplicial coordinates along that horocycle or cycle. Finally, we do the same for any edges associated to vanishing horocycles or cycles which are eliminated in the passage from the underlying fatgraph G to $\bar{G} = \pi(p)$. We then projectivize coordinates in comparability classes to obtain a well-defined point $\bar{x}(p) \in \mathcal{C}(f(p))$ for a nest $f(p)$. The following is a commutative diagram by construction:

$$\begin{array}{ccc} \mathcal{FS}(F) & \xrightarrow{\psi} & \hat{T}(F) \\ & \searrow \pi & \downarrow \chi \\ & & \mathcal{PG}(F) \end{array}$$

Lemma 5.14. *The map $\psi : \mathcal{FS}(F) \rightarrow \hat{T}(F)$ is a surjection.*

Proof. Given $(\bar{F}, \bar{x}) \in \hat{T}(F)$, consider $\bar{G} = \chi(\bar{F}, \bar{x})$. Following the same reasoning as in the proof of Lemma 5.2, we obtain a fatgraph G and associated filtered screen $\vec{\mathcal{E}}$ on G , where now we specify that for resolved horocycles and cycles both the levels and simplicial coordinates are such that we obtain a point $p \in \mathcal{C}(\vec{\mathcal{E}})$ for which $\psi(p) = (\bar{F}, \bar{x})$. \square

5.2. Cell decomposition and strata of $\mathcal{PG}(F)$. We now come to the first of our main results, which is an $MC(F)$ -invariant cell decomposition for $\mathcal{PG}(F)$ which follows directly from the above analysis.

Theorem 5.15. $\mathcal{PG}(F)$ admits an $MC(F)$ -invariant cell decomposition with the following properties:

- Cells are in one-to-one correspondence with isotopy classes of punctured fatgraphs with partial pairing \bar{G} supported by F , and for which $\vec{\mathcal{G}}_{\bar{G}}$ is a realizable partially oriented stratum graph.
- Each cell is a product of simplices, where projectivized simplicial coordinates on the component punctured fatgraphs give barycentric coordinates for the factor simplices.
- The union of cells with the same underlying partially oriented stratum graph is itself a product of projectivized partially decorated Teichmüller spaces, one for each irreducible component; we will refer to these as the strata of $\mathcal{PG}(F)$.
- The face relation for cells within a stratum is generated by collapsing edges (of punctured fatgraph components) with distinct endpoints, only one of which may be punctured.
- The face relation for cells in different strata is generated by vanishing of quasi recurrent subsets within an irreducible component F_i , followed by the map π .

Proof. We see from sub-sections 3.6 and 5.1, and in particular since χ is a surjection (Lemma 5.7), that $\mathcal{PG}(F)$ is composed of strata of products of projectivized partially decorated Teichmüller spaces, where a particular stratum has an underlying stratum T_σ from augmented Teichmüller space but with a choice of decorated subset \mathcal{P}_i of the punctures on each irreducible component F_i . We may label these strata by $\tilde{T}_{\cup \mathcal{P}_i}(F_\sigma)$, where the F_σ indicates the underlying stratum in augmented Teichmüller space, and the $\cup \mathcal{P}_i$ indicates the decorated punctures on each irreducible component F_i .

However, the subset of punctures which are decorated cannot be chosen arbitrarily but must be chosen consistently with a partially oriented stratum graph \bar{G} that supports a compatible nest. Nevertheless, as long as these conditions are satisfied, each stratum admits a cell decomposition, where cells are in one-to-one correspondence with isotopy classes of punctured fatgraphs with partial pairing compatible with the underlying partially oriented stratum graph. Each cell is a product of simplices, where simplicial coordinates on the component punctured fatgraphs give barycentric coordinates for the factor simplices. The face relation for cells within a stratum is generated by collapsing edges with distinct endpoints (only one of which may be punctured) in the punctured fatgraphs associated to irreducible components.

Moreover, since the face relation on $\mathcal{FS}(F)$ is also generated by coalescing of adjacent levels in filtered screens using the fact that $\mathcal{PG}(F)$ is then obtained by a quotient of $\mathcal{FS}(F)$ via the map π , we see that the face relation for cells in different strata in $\mathcal{PG}(F)$ is generated by way of $\mathcal{FS}_{\mathcal{P}_i}(F_i)$ and π for each irreducible component F_i in F_σ . More specifically, given a cell $\mathcal{C}(E_1) \times \dots \times \mathcal{C}(E_m)$ in a stratum $\tilde{T}_{\cup \mathcal{P}_i}(F_\sigma)$ where F_σ has irreducible components F_1, \dots, F_m , we consider a stable path \bar{X}_t which

only varies simplicial coordinates within one factor $\mathcal{C}(E_j)$ and whose limiting screen $\vec{\mathcal{E}}(\bar{X}_t)$ is of total level one. There will then be a limiting point $p(\bar{X}_t) \in \mathcal{C}(\vec{\mathcal{E}}(\bar{X}_t)) \subset \mathcal{FS}_{\mathcal{P}_i}(F_i)$ to which we may apply π to obtain a new labeled punctured fatgraph with partial pairing; this will yield a point in a new cell in a new stratum where the $\mathcal{C}(E_j)$ factor has been replaced by a (possible) product $\mathcal{C}(E_{j_1}) \times \cdots \times \mathcal{C}(E_{j_k})$. If the stable path simply yielded a new vanishing horocycle, we have moved from the stratum $\tilde{T}_{\cup \mathcal{P}_i}(F_\sigma)$ to a stratum $\tilde{T}_{\cup \mathcal{P}'_i}(F_\sigma)$, where the underlying stratum in augmented Teichmüller space has stayed fixed, but we have changed the collection of punctures which are decorated and hence have changed the underlying partially oriented stratum graph without changing the stratum graph. However, if the stable path yields a vanishing cycle or fatgraph, then we move to a stratum $\tilde{T}_{\cup \mathcal{P}'_i}(F_{\sigma'})$, where the underlying stratum in augmented Teichmüller space has itself changed.

Finally, we observe that the natural action of $MC(F)$ on $\mathcal{PG}(F)$ takes cells to cells, so the cell decomposition is $MC(F)$ -invariant. \square

We note briefly that a corollary of the theorem is that, as a set, indeed $\mathcal{PG}(F) = PG(F)$, with the added observation that conditions 1 and 2 of the definition of $PG(F)$ are precisely conditions ii and iii of Lemma 5.12.

We will establish a homotopy equivalence from $\mathcal{PG}(F)$ to $\bar{T}(F)$ much as we did in Proposition 4.23, where now we will generate a global flow using nests compatible with realizable partially oriented stratum graphs. In order to do so, we first need to analyze more closely the strata of $\mathcal{PG}(F)$, which are in one-to-one correspondence with realizable partially oriented stratum graphs.

Given a nest f on a stratum graph \mathcal{G} and an edge $n \in N$, recall from subsection 4.3 that we may always obtain a new nest $f_{\{n\}}$ by slightly lowering the comparability class of n so that it is isolated in its own level. In the case that f is compatible with a partially oriented stratum graph $\vec{\mathcal{G}}$ and n is oriented, the nest $f_{\{n\}}$ will be compatible with the partially oriented stratum graph $\vec{\mathcal{G}}(n)$ defined by the operation pictured in Figure 14. In that figure, the decrease in level of n results in x_1, \dots, x_k no longer being in $\text{Min}_{f_{\{n\}}}(v)$, and thus these edges become unoriented. In other words, all edges in $\text{Min}_f(v)$ except for n become unoriented in $\vec{\mathcal{G}}(n)$, and the orientations of all other edges remain the same.

On the other hand, if f is compatible with a partially oriented stratum graph $\vec{\mathcal{G}}$, and n is unoriented, then $f_{\{n\}}$ will be compatible with the same partially oriented stratum graph $\vec{\mathcal{G}}(n) = \vec{\mathcal{G}}$; in other words, the orientations do not change.

This motivates the following definition:

Definition 5.16. Given a realizable partially oriented stratum graph $\vec{\mathcal{G}}$ with underlying stratum graph \mathcal{G} , we say an edge $n \in N$ is *contractible* if

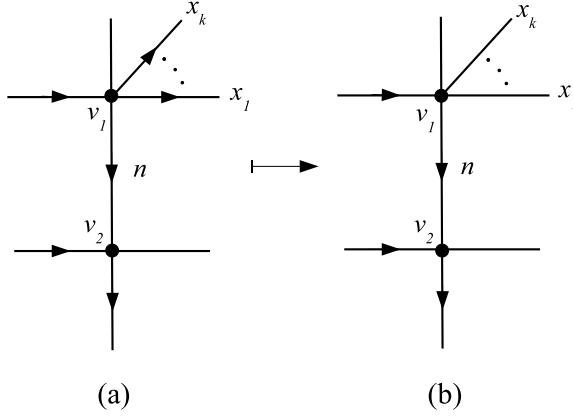


FIGURE 14. A partially oriented stratum graph $\vec{\mathcal{G}}$ changes to $\vec{\mathcal{G}}(n)$ when an oriented edge n has its level slightly decreased.

collapsing n in $\vec{\mathcal{G}}(n)$ maintaining all other orientations on all other edges in $\vec{\mathcal{G}}(n)$ yields a realizable partial orientation on the resulting stratum graph $\mathcal{G}_{\{n\}}$. We call such a sequence

$$\vec{\mathcal{G}} \rightarrow \vec{\mathcal{G}}(n) \rightarrow \vec{\mathcal{G}}_{\{n\}}$$

of partially oriented stratum graphs a *contracting sequence for n* ; contracting sequences for the cases where n is oriented and unoriented are shown in Figures 15 and 16.

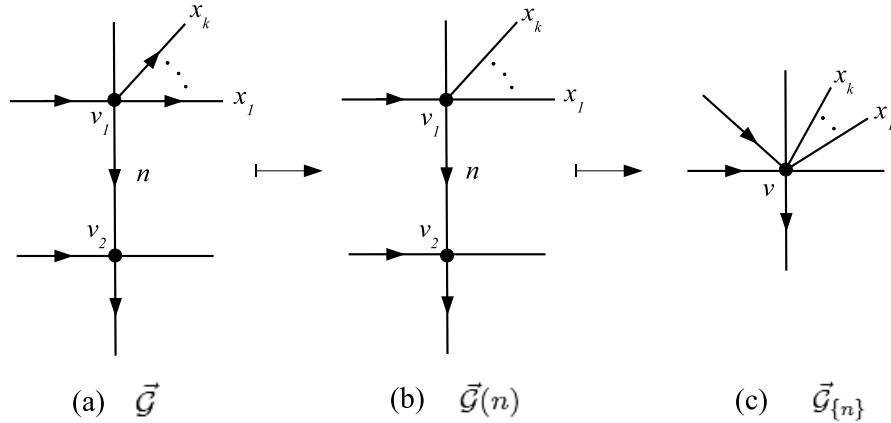


FIGURE 15. A contracting sequence for the oriented edge n .

Definition 5.17. Given a realizable partially oriented stratum graph $\vec{\mathcal{G}}$, an unoriented edge $n \in N$ is *essential* if its endpoints are distinct and there exists an oriented path in $\vec{\mathcal{G}}$ from one endpoint to the other. The unoriented edge n is said to be *inessential* otherwise.

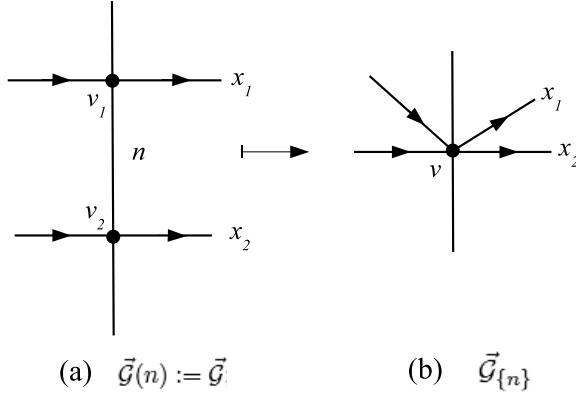


FIGURE 16. A contracting sequence for an unoriented edge.

The following lemma characterizes contractible edges:

Lemma 5.18. *If $\vec{\mathcal{G}}$ is a realizable partially oriented stratum graph and $n \in N$, then the following are equivalent:*

1. *n is contractible,*
2. *n is oriented or inessential,*
3. *n is collapsible for a nest f compatible with $\vec{\mathcal{G}}(n)$.*

Proof. To show that 1 is equivalent to 2, first suppose n is oriented and consider Figure 15. Since we began with a realizable partially oriented stratum graph in (a), the partially oriented stratum graph in (c) is also realizable as it satisfies i-iii of Lemma 5.12. Specifically, i is clearly true in (c) since it is true in (a); ii holds in (c) since v_1 was not the vertex satisfying item ii in (a); and iii holds in (c) since any oriented cycle in (c) would induce an oriented cycle in (a). Thus, n is contractible if n is oriented. If n is unoriented and essential, it is not contractible, for the induced partial orientation on the stratum graph $\mathcal{G}_{\{n\}}$ would have an oriented cycle, violating iii of Lemma 5.12. Conversely, if n is inessential, then the partial orientation on the stratum graph $\mathcal{G}_{\{n\}}$ will be realizable using the same argument as for the case of n oriented, and hence n is contractible.

To show that 1 is equivalent to 3, again first consider the case where n is oriented in $\vec{\mathcal{G}}$; we then know that n is contractible, and so we therefore must show that for any such $\vec{\mathcal{G}}(n)$ there exists a compatible nest f for which n is collapsible. To see this, consider parts (b) and (c) of Figure 15 and observe that in (c), $\vec{\mathcal{G}}_{\{n\}}$ supports a compatible nest f' where edges in $\text{Min}_{f'}(v)$ have some level j . We may then define f on $\vec{\mathcal{G}}(n)$ in (b) by simply assigning to edges other than n the levels of f' , and assigning to n the level $j + 1/2$. It is then evident that $[2f]$ is a nest compatible with $\vec{\mathcal{G}}(n)$, for which n is collapsible.

For the case where n is unoriented in $\vec{\mathcal{G}}$, refer to Figure 16 and observe that if n is collapsible for a compatible nest f on $\vec{\mathcal{G}}(n)$, then n satisfies 1 of Lemma

[4.9](#), and thus the levels of $x_1 \in \text{Min}_{f_{\{n\}}}(v_1)$ and $x_2 \in \text{Min}_{f_{\{n\}}}(v_2)$ in part (a) will be the same. Thus, the orientations on $\vec{\mathcal{G}}_{\{n\}}$ in part (b) are precisely the orientations for which $(f_{\{n\}})^{\{n\}}$ is compatible in the collapsing sequence for n , and hence these orientations are realizable, and n is contractible. Conversely, if n is contractible, then in (b), $\vec{\mathcal{G}}_{\{n\}}$ supports a compatible nest f' where edges in $\text{Min}_{f'}(v)$ have some level j . We may then define f as in the previous paragraph to obtain the compatible nest $[2f]$ for which n is collapsable. \square

We conclude with a lemma and corollary that applies the above lemma to the space $\mathcal{PG}(F)$:

Lemma 5.19. *Let $\vec{\mathcal{G}}$ be a realizable partially oriented stratum graph, let $n \in N$, and let $\vec{\mathcal{G}}(n)$ and $\vec{\mathcal{G}}_{\{n\}}$ be as in the contracting sequence for n . Then the stratum in $\mathcal{PG}(F)$ corresponding to $\vec{\mathcal{G}}$ shares a common boundary stratum with a stratum corresponding to some realizable partial orientation of $\mathcal{G}_{\{n\}}$ if and only if n is collapsable for a nest f compatible with $\vec{\mathcal{G}}(n)$. In this case, the realizable partial orientation of $\mathcal{G}_{\{n\}}$ may be assumed to be $\vec{\mathcal{G}}_{\{n\}}$, and the common boundary stratum may be assumed to be that corresponding to $\vec{\mathcal{G}}(n)$.*

We say that such a stratum corresponding to $\vec{\mathcal{G}}$ is *adjacent* to the union of strata corresponding to $\mathcal{G}_{\{n\}}$.

Proof. If n is collapsable for a nest f compatible with $\vec{\mathcal{G}}(n)$, then in $\hat{T}(F)$, the cell $\mathcal{C}(f)$ is adjacent to $\mathcal{D}(\sigma)$, where $\mathcal{G}_{\{n\}} = \mathcal{G}(\sigma)$. Furthermore, in the collapsing sequence that changes f on $\mathcal{G}(n)$ to $f^{\{n\}}$ on $\mathcal{G}_{\{n\}}$, the nest $f^{\{n\}}$ is compatible with $\vec{\mathcal{G}}_{\{n\}}$, as can be seen from Figure 15 in the case where n is oriented in $\vec{\mathcal{G}}(n)$ using reasoning as in Lemma 5.18 for the case where n is unoriented. Moreover, since by Lemma 5.14 ψ is surjective, decreasing the level of n in f in the corresponding collapsing sequence results in decreasing the level of the corresponding horocycle or cycle γ in filtered screens in the pre-image of ψ , which in turn results in the removal of γ from the boundary of these filtered screens. After applying π to these filtered screens and using the fact that $\chi \circ \psi = \pi$, we observe that indeed the stratum corresponding to $\vec{\mathcal{G}}(n)$ is a common boundary for the strata corresponding to $\vec{\mathcal{G}}$ and $\vec{\mathcal{G}}_{\{n\}}$.

Conversely, if the strata corresponding to $\vec{\mathcal{G}}$ and some realizable partial orientation of $\mathcal{G}_{\{n\}}$ share a common boundary stratum in $\mathcal{PG}(F)$, then in the pre-image of π there must be two filtered screens, $\vec{\mathcal{E}}$ and $\vec{\mathcal{E}}_{\{n\}}$, such that in $\vec{\mathcal{E}}$ there is a horocycle or cycle γ corresponding to n in $\partial \vec{\mathcal{E}}$ so that in decreasing the level of γ , one first moves to $\vec{\mathcal{E}}(n)$ in which γ is isolated (as a boundary cycle or horocycle) in its own level, and then one moves to $\vec{\mathcal{E}}_{\{n\}}$ and removes γ from the boundary of the filtered screen. As a result, in the image of ψ there will be a sequence of three nests, namely f , $f_{\{n\}}$, and $(f_{\{n\}})^{\{n\}}$, that

are compatible with $\vec{\mathcal{G}}$, $\vec{\mathcal{G}}(n)$, and $\vec{\mathcal{G}}_{\{n\}}$, respectively, and the result follows again using that $\chi \circ \psi = \pi$. \square

Corollary 5.20. *A stratum corresponding to $\vec{\mathcal{G}}$ is adjacent to the union of strata corresponding to $\mathcal{G}_{\{n\}}$ if and only if n is contractible in $\vec{\mathcal{G}}$.*

5.3. Equivariant homotopy equivalence. We require three lemmas before proving the main theorem that $\mathcal{PG}(F)$ is homotopy equivalent to augmented Teichmüller space, and this homotopy equivalence is equivariant for the action of $MC(F)$.

Lemma 5.21. *If $\vec{\mathcal{G}}$ is a realizable partially oriented stratum graph, and $n \in N$ is unoriented and essential, then there is a canonical orientation on n such that the result is a realizable partial orientation.*

Proof. Denote the two distinct endpoints of n by v_0 and v_1 and suppose that an oriented path which makes n essential is oriented from v_0 to v_1 . Any other oriented path connecting v_0 and v_1 must also run from v_0 to v_1 , for otherwise $\vec{\mathcal{G}}$ would contain an oriented cycle. \square

Lemma 5.22. *Let $\vec{\mathcal{G}}$ be a realizable partially oriented stratum graph and $n \in N$ unoriented and inessential. Suppose that the two vertex endpoints v_0, v_1 of n are such that the level of edges in $\text{Min}_{f_{\vec{\mathcal{G}}}}(v_0)$ is greater than the level of edges in $\text{Min}_{f_{\vec{\mathcal{G}}}}(v_1)$. Then orienting n from v_0 to v_1 yields a realizable partial orientation.*

Proof. The partial orientation results simply by decreasing the level of n in $f_{\vec{\mathcal{G}}}$ by one, thereby producing a compatible nest, so the partial orientation is realizable. \square

Lemma 5.23. *Let $\vec{\mathcal{G}}$ be a realizable partially oriented stratum graph with all edges contractible. Consider $f_{\vec{\mathcal{G}}}$, and suppose that the two vertex endpoints v_0, v_1 of any unoriented and inessential $n \in N$ are such that the level of edges in $\text{Min}_{f_{\vec{\mathcal{G}}}}(v_0)$ is equal to the level of edges in $\text{Min}_{f_{\vec{\mathcal{G}}}}(v_1)$. Let $\vec{\mathcal{G}}'$ be the realizable partially oriented stratum graph compatible with $(f_{\vec{\mathcal{G}}})^{\mathring{M}(f_{\vec{\mathcal{G}}})}$. Then all edges in $\vec{\mathcal{G}}'$ are contractible.*

Proof. We assume for contradiction that there exists an essential unoriented edge in $\vec{\mathcal{G}}'$ and call it n . As above, we assume that the oriented path connecting the two vertex endpoints of n goes from v_0 to v_1 . We have two cases, depending on whether n was oriented or unoriented in $f_{\vec{\mathcal{G}}}$.

In the former case, we find $n \notin \mathring{M}(f_{\vec{\mathcal{G}}})$, for otherwise, it would remain oriented. However, we then observe that decreasing the level of n by one in $(f_{\vec{\mathcal{G}}})^{\mathring{M}(f_{\vec{\mathcal{G}}})}$ still yields a nest, contradicting the maximality of $\mathring{M}(f_{\vec{\mathcal{G}}})$ as required.

In the latter case with n unoriented in $f_{\vec{\mathcal{G}}}$, since the level of $\text{Min}_{f_{\vec{\mathcal{G}}}}(v_0)$ is equal to the level of $\text{Min}_{f_{\vec{\mathcal{G}}}}(v_1)$, it must be that $n \notin \mathring{M}(f_{\vec{\mathcal{G}}})$, since for

$(f_{\vec{\mathcal{G}}})^{\dot{M}(f_{\vec{\mathcal{G}}})}$, the respective levels are unequal by assumption. However, again we obtain a contradiction as above that $\dot{M}(f_{\vec{\mathcal{G}}})$ is not maximal since we can still decrease the level of n and obtain a nest. \square

We now define a map $\phi : \mathcal{PG}(F) \rightarrow \bar{T}(F)$ by $\phi(\bar{G}) = \bar{F}(\bar{G})$ as in the proof of Lemma 5.7. The map ϕ is a surjection, and the pre-image of any stratum T_σ under ϕ is precisely the union of all the strata of $\mathcal{PG}(F)$ corresponding to realizable partial orientations of $\mathcal{G}(\sigma)$.

Theorem 5.24. *The map $\phi : \mathcal{PG}(F) \rightarrow \bar{T}(F)$ is an equivariant homotopy equivalence.*

This gives us immediately our main result:

Corollary 5.25. *$\mathcal{PG}(F)/MC(F)$ is homotopy equivalent to $\bar{\mathcal{M}}(F)$.*

Proof of Theorem 5.24. Given any $\bar{F} \in \bar{T}(F)$, consider a point $\bar{G} \in \phi^{-1}(\bar{F})$ with its underlying partially oriented stratum graph $\vec{\mathcal{G}}_0$. Apply Lemma 5.21 to all essential edges and orient them. Moreover, for each such new oriented edge, assign to its corresponding horocycle the minimum (deprojectivized) length of the original horocycles for \bar{G} in its respective component fatgraph while maintaining (deprojectivized) lengths of original horocycles for \bar{G} . We then projectivize decorations within components. This corresponds to homotopy within the fiber $\phi^{-1}(\bar{F})$ as we move from one stratum corresponding to a realizable partial orientation of \mathcal{G} to another; alternatively, it corresponds to decreasing by one the level of all essential edges in $f_{\vec{\mathcal{G}}_0}$. As a result, the new partially oriented stratum graph $\vec{\mathcal{G}}_1$ which we obtain has all contractible edges. The flow generated by performing this operation for each \bar{G} therefore preserves fibers of ϕ and is continuous within $\phi^{-1}(T_\sigma)$ for strata T_σ of $\bar{T}(F)$. Moreover, any edge which was contractible in $\vec{\mathcal{G}}_0$ is still contractible in $\vec{\mathcal{G}}_1$ by Corollary 5.20, and this means that the flow is continuous across adjacent $\phi^{-1}(T_\sigma)$ and $\phi^{-1}(T_{\sigma'})$. The image of the resulting global homotopy is supported by partially oriented stratum graphs all of whose edges are contractible.

Now define a second homotopy as follows: begin by taking a point \bar{G} with underlying partially oriented stratum graph $\vec{\mathcal{G}}_1$ having all contractible edges and apply Lemma 5.22 to all unoriented inessential edges for which the level of $\text{Min}_{f_{\vec{\mathcal{G}}_1}}(v_0)$ is greater than the level of $\text{Min}_{f_{\vec{\mathcal{G}}_1}}(v_1)$, orienting such edges; as before, assign to each new oriented edge the minimum length of the (deprojectivized) horocycle in its respective component fatgraph while maintaining (deprojectivized) lengths of original horocycles and finally projectivize decorations within components. As above, this corresponds to homotopy within the fiber $\phi^{-1}(\bar{F})$, and the resulting flow is continuous within and across fibers $\phi^{-1}(T_\sigma)$ since all contractible edges remain contractible in the resulting partially oriented stratum graph $\vec{\mathcal{G}}$.

We then apply the $\cdot^{\hat{M}}$ -operator to the image. Specifically, for each $f_{\vec{\mathcal{G}}}$, we decrease levels of edges by one to obtain $(f_{\vec{\mathcal{G}}})^{\hat{M}(f_{\vec{\mathcal{G}}})}$; to assign (deprojectivized) lengths of horocycles, we observe that at each vertex v of \mathcal{G} , the sets $\text{Min}_{f_{\vec{\mathcal{G}}}}(v)$ and $\text{Min}_{(f_{\vec{\mathcal{G}}})^{\hat{M}(f_{\vec{\mathcal{G}}})}}(v)$ will have non-empty intersection, and thus there will be a corresponding minimum length of decorated horocycle in this intersection. We can therefore assign this (deprojectivized) length to all new oriented edges for $(f_{\vec{\mathcal{G}}})^{\hat{M}(f_{\vec{\mathcal{G}}})}$ while maintaining (deprojectivized) lengths of horocycles in the intersection and finally projectivize within components. The resulting partially oriented stratum graph $\vec{\mathcal{G}'}$ for which $(f_{\vec{\mathcal{G}}})^{\hat{M}(f_{\vec{\mathcal{G}}})}$ is compatible will still have all edges contractible by Lemma 5.23 and will have a canonical minimal compatible nest $f_{\vec{\mathcal{G}'}}$.

By repeated application of Lemmas 5.22 and 5.23, as we move through strata corresponding to different realizable partial orientations $\vec{\mathcal{G}}$, the levels of edges for the canonical minimal nest $f_{\vec{\mathcal{G}}}$ are monotonically decreasing. The flow thus generated is evidently continuous within and across fibers $\phi^{-1}(T_\sigma)$ as all edges remain contractible. There results a global homotopy of $\mathcal{PG}(F)$ with image in the strata of $\mathcal{PG}(F)$ corresponding to partially oriented stratum graphs compatible with f_σ , the minimal nests for each T_σ , for every $\bar{F} \in T_\sigma$. This flow is $MC(F)$ -equivariant, and since these strata corresponding to partially oriented stratum graphs compatible with f_σ are all products of projectivized partially decorated Teichmüller spaces, they are themselves contractible. \square

6. NEW CYCLES ON $\bar{\mathcal{M}}(F)$

Fix a surface $F = F_g^s$. Let us here just give a simple construction of cycles on $\bar{\mathcal{M}}(F)$ which includes certain odd-degree cycles. A more extensive study of the (co)homology of $\bar{\mathcal{M}}(F)$ using our cell decomposition will be taken up elsewhere.

In general using the formalism of filtered screens and punctured fatgraphs with partial pairings to probe the homology of $\bar{\mathcal{M}}(F)$, one must pay special attention to the dimensions. For example, consider a filtered screen (L^0, L^1) of level one, so that $L^{\geq 0}$ is a q.c.d of F containing the proper quasi recurrent set L^1 and let G^0 and $G^1 \subset G^0$ denote the respective dual punctured fatgraphs. It may be as in Figure 8 that a number of edges of G^0 combine to form a single edge of G^1 , in which case there is a concomitant drop of dimension from the cell in $\mathcal{FS}(F)$ to $\mathcal{PG}(F)$. On the other hand, it may be that each vertex of G^1 has valence different from two in G^1 , in which case there is no such drop in dimension by construction.

Definition 6.1. The *valence spectrum* of an arbitrary punctured fatgraph G is the tuple v_1, v_2, \dots , where v_k is the number of vertices of valence k , for $k \geq 1$. The *length spectrum* of G is the tuple ℓ_1, ℓ_2, \dots , where ℓ_j is the number of boundary components of length j , for $j \geq 1$ and the length is the

the number of edges traversed by an efficient representative counted with multiplicity.

Given the valence spectrum v_1, v_2, \dots of a punctured fatgraph G in F with p punctured and u unpunctured vertices, we have

$$\chi(G) = -p - \frac{1}{2} \sum_{i \geq 3} (i-2)v_i,$$

whereas in terms of the length spectrum ℓ_1, ℓ_2, \dots of G , we find

$$\chi(G) = u - \frac{1}{2} \sum_{j \geq 1} j\ell_j.$$

Definition 6.2. A *rose* is a punctured fatgraph with at least one edge and a single vertex, which must be punctured if there is exactly one edge, and a rose is called *odd* provided each of its boundary components has odd length, i.e., in the length spectrum, $\ell_j = 0$ for j even.

An *odd-valence fatgraph* is a punctured fatgraph with at least one edge so that each possibly punctured vertex has odd valence, i.e., in the valence spectrum, $v_k = 0$ for k even, provided furthermore that each univalent vertex is punctured.

An *odd-length fatgraph* is a punctured fatgraph whose dual is an odd-valence fatgraph (where the dual is taken in the closed surface of the same genus interchanging the roles of vertices and boundary cycles) provided furthermore that each edge is either a loop or has both endpoints punctured.

In particular, notice that any edge hence any subset of edges of an odd-length fatgraph G is recurrent, and the boundary operator of filtered screens in $\bar{\mathcal{M}}(F)$ restricts to $\mathcal{C}(G)$ giving the usual one on the first barycentric subdivision of the set of edges of G partially ordered by inclusion. Examples of odd-length fatgraphs include odd roses or indeed the dual to any odd-valence fatgraph where each vertex is taken to be punctured.

To define our cycles, specify a tuple $\vec{v}^\mu = (v_1^\mu, v_2^\mu, \dots)$ of putative valence spectra for odd-valence fatgraphs as well as a tuple $\vec{\ell}^\nu = (\ell_1^\nu, \ell_2^\nu, \dots)$ of putative length spectra for odd-length fatgraphs, where $\mu = 1, \dots, m$ and $\nu = 1, \dots, n$. A necessary condition for there to be some partial pairing on these component fatgraphs which is supported by F is

$$\begin{aligned} (*) \quad & 2g - 2 + s = \#\{\text{punctured vertices in odd-valence fatgraphs}\} \\ & - \#\{\text{unpunctured vertices in odd-length fatgraphs}\} \\ & + \frac{1}{2} \sum_{\mu} \sum_{i \geq 3} (i-2)v_i^\mu + \frac{1}{2} \sum_{\nu} \sum_{j \geq 1} j\ell_j^\nu \end{aligned}$$

according to Definition 1.2.

Proposition 6.3. Fix $\vec{v}^1, \dots, \vec{v}^m$ and $\vec{\ell}^1, \dots, \vec{\ell}^n$ as above and consider the collection of cells in $\mathcal{PG}(F)$ corresponding to punctured fatgraphs with partial pairing comprised of m odd-valence fatgraph (unordered) components with

respective valence spectra $\vec{v}^1, \dots, \vec{v}^m$ and n odd-length fatgraph (unordered) components with respective length spectra $\vec{\ell}^1, \dots, \vec{\ell}^n$ satisfying condition (*). These cells admit orientations rendering the closure of their union a non-trivial cycle in $\mathcal{PG}(F)/MC(F)$.

Proof. The collection of odd-valence fatgraphs in a fixed surface F with fixed valence spectrum can be oriented to produce cycles of non-compact support in $\mathcal{M}(F)$, cf. [8, 19] and see [13] for a nice detailed presentation. As shown independently in [6, 7] and [14], the polynomial algebra in cohomology spanned by duals of these “combinatorial classes” agrees with the polynomial algebra spanned by the MMM classes on $\mathcal{M}(F)$. Furthermore, it is well-known [1] that the MMM classes extend to $\bar{\mathcal{M}}(F)$, from which we conclude that the combinatorial classes likewise extend. It follows that the closure of the union of these cells gives a cycle in compactified $\bar{\mathcal{M}}(F) \approx \mathcal{PG}(F)/MC(F)$ in this case.

Given an odd-length fatgraph G in a fixed surface F , the dual is an odd-valence fatgraph admitting the orientation just discussed, which in turn orients the cell corresponding to G . Removing edges of the former corresponds to contracting edges of the latter, and the standard cancellations described in [13] ensure that the induced orientations render the union of cells corresponding to odd-length fatgraphs with fixed length spectrum a cycle in $\mathcal{PG}(F)/MC(F)$ in this dual case. Non-triviality of the MMM classes implies non-triviality of these dual classes as well. \square

We conjecture that these classes, certain of which occur in odd degree, are all homologically non-trivial.

Example 6.4. To construct odd-length fatgraphs in each genus, start with a rose R with $F(R) = F_g^1$, whose non-zero length spectrum is $\ell_{4g} = 1$, and add a single petal P to R whose endpoints are consecutive. The resulting rose $P \cup R$ has non-zero length spectrum $\ell_1 = 1, \ell_{4g+1} = 1$ and is thus an odd-length fatgraph of genus g with two boundary components. To now exhibit a cell in \mathcal{PG} of odd codimension of the type described in Proposition 6.3, take any odd-valence fatgraph G of type F_h^1 and then enlarge G by adding an edge connecting an interior point of some edge of G to the vertex of R in between the endpoints of P to produce a new fatgraph containing $G \cup P \cup R$. The cell for this fatgraph is of even codimension in $\mathcal{PG}(F_{g+h}^2)$, and the cell corresponding to the filtered screen $(G, P \cup R)$ projects to an odd codimension cell in $\mathcal{PG}(F_{g+h}^2)$.

Example 6.5. To exhibit a cell of odd codimension in $\mathcal{PG}(F_{g+4}^1)$ of the type described in Proposition 6.3, first take an odd rose $P \cup R$ for F_g^2 as described in the previous example but now with a punctured vertex, and also add an extra punctured vertex at an interior point of one of the petals in R . The result is an odd-length fatgraph which we still denote by $P \cup R$ but now of type F_g^4 with two boundary components and two punctured vertices. Take

also a uni-trivalent fatgraph G for F_0^5 having four univalent vertices and one boundary component, and attach two of the univalent vertices to an interior point of P , one along each of the two boundary components of $P \cup R$. Finally, add two cycles γ_1 and γ_2 each having a single edge, where all edges incident on the central punctured vertex of $P \cup R$ attach on one side of the single vertex of γ_1 and the third univalent vertex of G attaches on the other side; likewise, the two edges of the bivalent punctured vertex in R attach on one side of the single vertex of γ_2 and the fourth univalent vertex of G attaches on the other side. The resulting fatgraph represents F_{g+4}^1 and has a single even-valent vertex, and hence its cell has odd codimension in $\mathcal{PG}(F_{g+4}^1)$. Moreover, the cell corresponding to the filtered screen $(G, P \cup R, \gamma_1 \cup \gamma_2)$ projects to an odd codimension cell in $\mathcal{PG}(F_{g+4}^1)$, as the vanishing of γ_1 and γ_2 represent an increase in codimension of one each, respectively, the vanishing fatgraph $P \cup R$ represents an additional increase of codimension by one, and finally the separate projectivization in the two component fatgraphs completes the increase in codimension to a total of four.

7. CONCLUDING REMARKS

One reason that a cell decomposition for $\bar{\mathcal{M}}$ has remained a compelling open problem is that a number of computations including [6, 7, 8, 14] might be repeated more completely and coherently in this context. For example in what appears to be an eminently feasible calculation, rather than appeal to these results as we did in the previous section, one should directly compute the cycles in $\bar{\mathcal{M}}$ which restrict to the combinatorial classes.

The computation of the (co)homology of $\bar{\mathcal{M}}$ is of course an outstanding open problem which we hope and expect to be informed by the cell decomposition we have derived here.

APPENDIX A. $\mathcal{FS}(F)$ AS A BLOW UP OF THE ARC COMPLEX OF F

Recall [20, 21] the *arc complex* $\text{Arc}_{\mathcal{P}}(F)$ of F based at \mathcal{P} is the simplicial complex with a k -simplex for each isotopy class of $k + 1$ ideal arcs, all based at a subset of punctures \mathcal{P} , which are pairwise disjointly embedded in F except perhaps at their endpoints and no two of which are parallel, and the face relation is induced by inclusion of arc families. Each such simplex has its natural barycentric coordinates, and so $\text{Arc}_{\mathcal{P}}(F)$ can be defined more abstractly as the collection of all projectively weighted suitable families of arcs in F . If \mathcal{P} consists of all the punctures, then we simply write $\text{Arc}(F)$.

In this section, we shall show that $\mathcal{FS}_{\mathcal{P}}(F)$ is a real blow up of $\text{Arc}_{\mathcal{P}}(F)$ by producing a sequence of maps which embed progressively larger collections of cells of filtered screens in F into successive retractions of $\text{Arc}_{\mathcal{P}}(F)$, each of which is obtained by blowing up certain faces. This finite sequence of maps terminates in a final embedding of $\mathcal{FS}_{\mathcal{P}}(F)$ into a blow-up of $\text{Arc}_{\mathcal{P}}(F)$; by construction, the topology will agree with Definition 3.11 and Propositions 3.21 and 3.23. Insofar as $P\tilde{T}_{\mathcal{P}}(F) \subset \text{Arc}_{\mathcal{P}}(F)$ by identifying projective

simplicial coordinates on q.c.d.'s with abstract projective weights on arc families, $\mathcal{FS}_P(F)$ provides a cellular bordification of $P\tilde{T}_P(F)$.

This appendix is dedicated to proving the following result:

Theorem A.1. *$\mathcal{FS}_P(F)$ is a cellular bordification of $P\tilde{T}_P(F)$ obtained from $\text{Arc}_P(F)$ by blowing up all ideal faces corresponding to $E - H$, where E is a q.c.d. of F and $H \subsetneq E$ is quasi recurrent.*

Proof. We will prove that $\mathcal{FS}(F)$ is a cellular bordification of $P\tilde{T}(F)$ obtained from $\text{Arc}(F)$ by blowing up all ideal faces corresponding to $E - H$, where E is an i.c.d. of F and $H \subsetneq E$ is recurrent. The proof will then apply verbatim for the general case of $\mathcal{FS}_P(F)$ by simply replacing everywhere the words “recurrent” by “quasi recurrent” and “i.c.d.” by “q.c.d.”.

For the purposes of this proof, we make the following definitions:

Let E be an i.c.d. of F and let H be a proper recurrent subset of E . We say that H is a *1-minimal recurrent subset* of E if H contains no proper recurrent subsets. Having recursively defined n -minimal recurrent subsets, we say that H is an *$(n+1)$ -minimal recurrent subset* of E if all proper recurrent subsets of H are k -minimal recurrent for $1 \leq k \leq n$ and there is at least one k -minimal recurrent subset for each such k . A filtered screen $\vec{\mathcal{E}}$ is *based on a k -minimal recurrent subset H* if H is k -minimal recurrent and is comprised of precisely those edges of greatest level in $\vec{\mathcal{E}}$. Define $\mathcal{FS}^k(F)$ to be the set of all cells associated to filtered screens that are based on a k -minimal recurrent subset. If n is the maximum value for all such k (so $n \leq 3g - 3 + 2s$), then define $\mathcal{FS}^{n+1}(F)$ to be the union of those cells in $\mathcal{FS}(F)$ associated to all filtered screens $\vec{\mathcal{E}} = (\{E\})$ corresponding to i.c.d.'s E of F .

We shall construct a sequence of embeddings Φ_0, \dots, Φ_n , where the domain of Φ_k is

$$\bigsqcup_{\substack{C(\vec{\mathcal{E}}) \in \mathcal{FS}^i(F) \\ (n-k+1) \leq i \leq n+1}} C(\vec{\mathcal{E}})$$

and the range is given by successive retractions of $\text{Arc}(F)$ corresponding to blowing up progressively more faces. The construction terminates with the map Φ_n , whose domain is $\mathcal{FS}(F)$ and whose image will be a blow up of $\text{Arc}(F)$.

Step 0: There is a natural map of cells

$$\Phi_0 : \bigsqcup_{C(\vec{\mathcal{E}}) \in \mathcal{FS}^{n+1}(F)} C(\vec{\mathcal{E}}) \hookrightarrow \text{Arc}(F)$$

sending projective simplicial coordinates to barycentric coordinates on arc families and embedding $P\tilde{T}(F)$ into $\text{Arc}(F)$. There are cells in $\text{Arc}(F)$ corresponding to arc families whose complementary regions have non-trivial topology which are not in $\text{Im } \Phi_0$. We may nevertheless use our familiar notation $C(E')$ for the open cell in $\text{Arc}(F)$ corresponding to such an arc family E' , whose closure in $\text{Arc}(F)$ will be denoted $\bar{C}(E')$.

Step 1: Let H be an n -minimal recurrent subset of an i.c.d. E . Thus, $\bar{\mathcal{C}}(E)$ is the *join* $\bar{\mathcal{C}}(E - H) * \bar{\mathcal{C}}(H)$ of $\bar{\mathcal{C}}(E - H)$ and $\bar{\mathcal{C}}(H)$ in $\text{Arc}(F)$; that is, points in $\bar{\mathcal{C}}(E)$ can be written as convex combinations $(1 - t)x_0 + tx_1$ for $x_0 \in \mathcal{C}(E - H)$, $x_1 \in \mathcal{C}(H)$, and $0 \leq t \leq 1$. For fixed x_0, x_1 and varying $0 \leq t \leq 1$, $(1 - t)x_0 + tx_1$ gives a path from x_0 to x_1 , where t is called the *join parameter* for $\mathcal{C}(E)$ corresponding to H .

Now, for $t_* > 0$ small, there is a 1-parameter family of maps r_s , parametrized by s , for $0 \leq s \leq t_*$, given by

$$t \mapsto (1 - s)t + s$$

which yields a retraction of $\mathcal{C}(E)$ onto

$$D(E) = \{(1 - t)x_0 + tx_1 \mid x_0 \in \mathcal{C}(E - H), x_1 \in \mathcal{C}(H), t_* < t < 1\}$$

in $\text{Arc}(F)$. We shall call this a *retraction of $\mathcal{C}(E)$ along the join parameter for $\mathcal{C}(E)$ corresponding to H* . It induces a retraction on the image of Φ_0 , that is, we have a map from $\mathcal{C}(\{E\})$ in $\mathcal{FS}^{n+1}(F)$ to $D(E)$ given by $r_{t_*} \circ \Phi_0$ and shall denote this map Φ_1 .

Now, the face

$$\{(1 - t_*)x_0 + t_*x_1 \mid x_0 \in \mathcal{C}(E - H), x_1 \in \mathcal{C}(H)\}$$

is a product simplex which is in fact a copy of $\mathcal{C}(E - H) \times \mathcal{C}(H)$ and can be parametrized with coordinates from the two simplex factors. Thus, this retraction yields a blow up, namely, of the face of $\mathcal{C}(E)$ corresponding to $\mathcal{C}(E - H)$ to $\mathcal{C}(E - H) \times \mathcal{C}(H)$. Furthermore, there is a unique filtered screen $\vec{\mathcal{E}}$ for which $L^0 = E - H$ and $L^1 = H$ are the edges of greatest level, and we extend the map Φ_1 to take $\mathcal{C}(\vec{\mathcal{E}})$ in $\mathcal{FS}^n(F)$ to this copy of $\mathcal{C}(E - H) \times \mathcal{C}(H)$ in a way which respects coordinates. Thus, in the image of Φ_1 for fixed x_0, x_1 , the path $(1 - t)x_0 + tx_1$ in $\Phi_1(\mathcal{C}(\{E\}))$ converges to the point (x_0, x_1) in $\Phi_1(\mathcal{C}(\vec{\mathcal{E}}))$, as it should according to Proposition 3.21 and Definition 3.11.

Note that Φ_1 on $\mathcal{C}(\{E\})$ restricts to Φ_1 on $\mathcal{C}(\{E'\})$ for any i.c.d. $E' \subset E$, so we can perform retractions and blow ups for all n -minimal H contained in an i.c.d. of F so that these retractions and blow ups agree on common faces. This yields a global retraction of $\text{Arc}(F)$ onto a subspace $\text{Arc}_1(F)$, which corresponds to blowing up all such faces to product simplices, and we have a well-defined embedding

$$\Phi_1 : \bigsqcup_{\substack{\mathcal{C}(\vec{\mathcal{E}}) \in \mathcal{FS}^i(F) \\ n \leq i \leq n+1}} \mathcal{C}(\vec{\mathcal{E}}) \hookrightarrow \text{Arc}_1(F)$$

In particular for $n \neq 1$, there are cells in $\text{Arc}_1(F)$ that are not in $\text{Im } \Phi_1$.

Step k+1: At the general iteration, we have the embedding

$$\Phi_k : \bigsqcup_{\substack{\mathcal{C}(\vec{\mathcal{E}}) \in \mathcal{FS}^i(F) \\ (n-k+1) \leq i \leq n+1}} \mathcal{C}(\vec{\mathcal{E}}) \hookrightarrow \text{Arc}_k(F),$$

where $\text{Arc}_k(F)$ is a retract of $\text{Arc}(F)$ obtained by blowing up faces to product simplices.

Let H be an $(n - k)$ -minimal recurrent subset of an i.c.d. E . Just as in the $k = 1$ step for the $\mathcal{C}(E - H)$ face of $\mathcal{C}(E)$, we may perform a blow up using a retraction r_s along the join parameter t for $\mathcal{C}(E)$ corresponding to H to produce the new map $\Phi_{k+1} = r_{t_*} \circ \Phi_k$ when restricted to $\mathcal{C}(\{E\})$. Furthermore, if $\vec{\mathcal{E}}$ is the unique filtered screen for which $L^0 = E - H$ with $L^1 = H$ of greatest level, then we extend the map Φ_1 to take $\mathcal{C}(\vec{\mathcal{E}})$ to the new blown up face

$$\{(1 - t_*)x_0 + t_*x_1 | x_0 \in \mathcal{C}(E - H), x_1 \in \mathcal{C}(H)\}$$

However, there is something new at this general iteration since a face of $\mathcal{C}(E)$ in $\text{Arc}_k(F)$ may already have been blown up. More specifically, suppose

$$H \subset E^j \subset E^{j-1} \subset \cdots \subset E^1 \subset E,$$

where each of the E^i is recurrent. It follows that in $\text{Arc}_k(F)$, $\mathcal{C}(E)$ has a face of the form

$$\mathcal{C}(E - E^1) \times \mathcal{C}(E^1 - E^2) \times \cdots \times \mathcal{C}(E^{j-1} - E^j) \times \mathcal{C}(E^j)$$

which corresponds to the filtered screen where the level sets of edges are precisely

$$L^0 = E - E^1, \quad L^1 = E^1 - E^2, \quad \dots \quad L^j = E^j.$$

In this cell, we also perform a blow up on the complement of H .

To understand this, note that points in this cell can be written as convex linear combinations

$$(1 - t)(\vec{x}, x_0) + t(\vec{x}, x_1),$$

for $0 < t < 1$, where

$$\vec{x} \in \mathcal{C}(E - E^1) \times \mathcal{C}(E^1 - E^2) \times \cdots \times \mathcal{C}(E^{j-1} - E^j),$$

with

$$x_0 \in \mathcal{C}(E^j - H) \quad \text{and} \quad x_1 \in \mathcal{C}(H).$$

This join parameter t then also yields a blow up, namely, the face of

$$\mathcal{C}(E - E^1) \times \mathcal{C}(E^1 - E^2) \times \cdots \times \mathcal{C}(E^{j-1} - E^j) \times \mathcal{C}(E^j)$$

corresponding to

$$\mathcal{C}(E - E^1) \times \mathcal{C}(E^1 - E^2) \times \cdots \times \mathcal{C}(E^{j-1} - E^j) \times \mathcal{C}(E^j - H)$$

is blown up to

$$\mathcal{C}(E - E^1) \times \mathcal{C}(E^1 - E^2) \times \cdots \times \mathcal{C}(E^{j-1} - E^j) \times \mathcal{C}(E^j - H) \times \mathcal{C}(H).$$

There is then a unique filtered screen $\vec{\mathcal{E}}$ with level sets

$$L^0 = E - E^1, \quad L^1 = E^1 - E^2, \quad \dots \quad L^j = E^j - H, \quad L^{j+1} = H,$$

and we extend the map Φ_{k+1} to take the cell corresponding to $\vec{\mathcal{E}}$ to

$$\mathcal{C}(E - E^1) \times \mathcal{C}(E^1 - E^2) \times \cdots \times \mathcal{C}(E^{j-1} - E^j) \times \mathcal{C}(E^j - H) \times \mathcal{C}(H)$$

in a manner that respects coordinates.

As before, this is consistent with blow ups on cells associated to filtered screens that are obtained by removing edges. We can therefore compatibly perform retractions and blow ups for all $(n - k)$ -minimal H contained in an i.c.d. E of F . This yields a global retraction of $\text{Arc}_k(F)$ to $\text{Arc}_{k+1}(F)$, which corresponds to blowing up faces to product simplices, and we have a well-defined embedding

$$\Phi_{k+1} : \bigsqcup_{\substack{\mathcal{C}(\vec{\mathcal{E}}) \in \mathcal{FS}^i(F) \\ (n-k) \leq i \leq n+1}} \mathcal{C}(\vec{\mathcal{E}}) \hookrightarrow \text{Arc}_{k+1}(F).$$

The sequence of embeddings terminates with $n - k = 1$, when every ideal face of $P\tilde{T}(F)$ in $\text{Arc}(F)$ has been blown up yielding $\mathcal{FS}(F)$ as a cellular bordification of $P\tilde{T}(F)$. \square

REFERENCES

- [1] E. Arbarello and M. Cornalba, *Combinatorial and algebro-geometric cohomology classes on the moduli spaces of curves*, J. Algebraic Geom. **5** (1996), 705–749.
- [2] L. Bers, *Spaces of degenerating Riemann surfaces*, 43–55, Ann. of Math. Studies No. 79, Princeton Univ. Press (1974).
- [3] J. Brock, *The Weil-Petersson metric and volumes of 3-dimensional hyperbolic convex cores*, J. Amer. Math. Soc. **16** (2003), 495–535.
- [4] D. B. A. Epstein and R. C. Penner, *Euclidean decompositions of noncompact hyperbolic manifolds*, J. Diff. Geom. **27** (1988), 67–80.
- [5] J. Harer and D. Zagier, *The Euler characteristic of the moduli space of curves* Invent. Math. **85** (1986), 457–485.
- [6] K. Igusa, *Combinatorial Miller-Morita-Mumford classes and Witten cycles*, Alg. Geom. Top. **4** (2004), 473–520.
- [7] —, *Graph cohomology and Kontsevich cycles*, Topology **43** (2004), 1469–1510.
- [8] M. Kontsevich, *Intersection theory on the moduli space of curves and the matrix Airy function*, Comm. Math. Phys. **147** (1992), 1–23.
- [9] E. Looijenga, *Cellular decompositions of compactified moduli spaces of pointed curves*, The moduli space of curves (Texel Island, 1994), Birkhäuser, Boston (1995), pp. 369–400.
- [10] I. Madsen and M. Weiss, *The stable mapping class group and stable homotopy theory* in European Congress of Mathematics, Stockholm 2004, Eur. Math. Soc., Zürich, 2005, 283–307.
- [11] G. McShane and R.C. Penner, *Stable curves and screens on fatgraphs*, in Proceedings of the Research Institute for Mathematics, Kyoto University (2007).
- [12] E. Miller, *The homology of the mapping class group*, J. Differential Geom. **24** (1986), 1–14.
- [13] G. Mondello, Combinatorial classes on the moduli space of curves are tautological, Scuola Normale Superiore di Pisa (2003).
- [14] G. Mondello, *Combinatorial classes on the moduli space of Riemann surfaces are tautological*, Int. Math. Res. Not. (2004), no. 44, 2329–2390.
- [15] S. Morita, *Characteristic classes of surface bundles*, Invent. Math. **90** (1987), 551–577.
- [16] D. Mumford, *Towards an enumerative geometry of the moduli space of curves* in: Arithmetic and geometry **2**, Progr. Math. **36**, Birkhäuser, Boston, MA, 1983. 271–328.
- [17] R.C. Penner, *The decorated Teichmüller space of punctured surfaces*, Commun. Math. Phys. **113** (1987), 299–339.

- [18] —, *Perturbative series and the moduli space of Riemann surfaces*, J. Differential Geom. **27** (1988), 35–53.
- [19] —, *The Poincaré dual of the Weil-Petersson Kähler form*, Comm. Anal. Geom. **1** (1993), 43–70.
- [20] —, *The structure and singularities of quotient arc complexes*, J. Top. **1** (2008), 527–550.
- [21] —, *Decorated Teichmüller theory*, European Mathematical Society, Zürich (2011).
- [22] U. Tillmann, *Strings and the stable cohomology of mapping class groups* in *Proceedings of the International Congress of Mathematicians*, Higher Ed. Press, Beijing, 2002, 447–456.
- [23] S. Wolpert, *Geometry of the Weil-Petersson completion of Teichmüller space*, Surv. Diff. Geom. **8** (2002), 357–393.

CENTRE FOR QUANTUM GEOMETRY OF MODULI SPACES, AARHUS UNIVERSITY
E-mail address: `dlafout@qgm.au.dk`

CENTRE FOR QUANTUM GEOMETRY OF MODULI SPACES, AARHUS UNIVERSITY, AND
 DEPARTMENTS OF MATHEMATICS AND THEORETICAL PHYSICS, CALTECH
E-mail address: `rpenner@qgm.au.dk` or `rpenner@caltech.edu`